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Generalized Eulerian Numbers and Multiplex Juggling Sequences

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Abstract

We consider generalizations of both juggling sequences and non-attacking rook placements. We demonstrate the important connection between these objects, and also propose a generalization of the Eulerian numbers. These generalizations give rise to several interesting counting problems, which we explore.

1 History of Mathematics of Juggling

Juggling as an activity has been around for the last 4000 years. There are many ancient records, both written documents and depictions, of people performing amazing juggling feats (for more on this, see [19]). Siteswap, the first mathematical way to describe juggling patterns, was formalized around 1985. Several papers appeared in the earlier years which investigated some of the possible ways to express juggling sequences, but they lacked consistency. In [20], published in 1982, Walker uses the idea of throw heights to invent new juggling patterns, even though the formal concept of a "throw height" had not yet been fully explored in the juggling community. Two years later, Buhler and Graham explored the physical dynamics of several commonly known juggling patterns [4]. Their paper considers many parameters of juggling. Several of these parameters, such as dwell time of a ball in one's hand, do not reappear in any of these authors' future juggling papers, and we assume they found the analysis of all these different factors too complicated to be useful.

In 1985, though, the juggling community became more cohesive, as three groups separately developed the beginnings of the modern mathematics of juggling [19]. These juggling founders included Paul Klimek in Santa Cruz, California, Bruce Tiemann and Bengt Magnusson at Caltech, and Colin Wright at Cambridge University. A record of some early emails from Magnusson about his siteswap system, as well as his computer program to generate siteswaps, can be found at [14]. Wright provides a thorough introduction to the concept of siteswap in both [11] and [21]. Most of this work was done mainly for jugglers. Developing a formal system to express juggling patterns allows jugglers to smoothly communicate about specific patterns as well as to invent new patterns.

Joe Buhler and Ron Graham, who are both passionate about juggling and mathematics, were the first to explore the mathematics of juggling more deeply. Armed with the structure of siteswaps, both Buhler and Graham wrote many papers, most using combinatorics to count juggling patterns with certain constraints. Siteswaps have a connection to permutations, which the authors exploit heavily. In 1994, Buhler and Graham, along with Wright and David Eisenbud, wrote *Juggling Drops and Descents*, a paper that provided the foundation for many future juggling papers [3]. It took many of the properties known about siteswap and translated them into a more mathematical setting. Furthermore, it was the first paper to highlight the connection between siteswap sequences and the Eulerian numbers. This connection was developed by showing a link between permutation drops and juggling sequences. Then, by forming a bijection between permutation drops and descents, we can apply methods for counting permutation descents to juggling sequences. The Eulerian numbers are a well-known object that counts permutations with a certain number of descents, and this link provides a convenient way to count one basic form of juggling sequences.

The first two sections of this paper give background information about the mathematics of juggling. Section 2 gives an in-depth introduction on siteswap notation and demonstrates how siteswap allows us to connect juggling sequences to rook placements. Section 3 explain the connection between juggling sequences and the Eulerian numbers, as is first discussed by Buhler et al. in [3]. Section 4 introduces the work done by the my group at the 2015 REU at Iowa State, which counted multiplex juggling sequences by connecting them to a generalized version of the Eulerian numbers. This section primarily defines the generalized Eulerian numbers and proves some basic properties of this family of numbers. At the REU, I was in charge of the proof in subsection 3. Section 5, which is original work, explores a recurrence relation of objects similar, but simpler, to what we are most interested in counting. Since finding a recurrence for the generalized Eulerian number was one proposed goal of the project, a related recurrence could potentially offer some insights. Section 6 is again work from the REU and describes a method of counting mutliplex juggling sequences. Sections 7 and 8 are fully original work. Section 7 explores one way to perhaps form a recurrence for the generalized Eulerian numbers, and describes the next steps that would be necessary for a nice, closed recurrence. Section 8 investigates whether there is a nice way to make the counting method from section 6 more efficient.

2 Basics of Math and Juggling

2.1 Assumptions and Siteswap Notation

When juggling, we make several assumptions. First, we assume we have been juggling forever and will continue to juggle forever. This prevents any complications involving starting and ending a juggling sequence. Next, we assume that we juggle to a fixed beat, as though a metronome is dictating the throws. We assume that, if we catch a ball on a beat, we immediately throw the ball again on that same beat. This implies that a ball will never simply rest in a juggler's hand; theoretically it is always in flight. (See Example 2 for an exception when we consider the physical interpretation of a juggling sequence.) The mathematics of juggling assumes everyone is a perfect juggler; that is, as long as there are not too many balls landing at the same beat, we will perfectly catch and throw each. The objects involved in juggling do not affect the mathematics, and neither does the path of flight.

In standard juggling, we describe the action on the i-th beat in terms of the *throw height*, t_i, where t_i is a nonnegative integer. If $t_i = 0$, no action occurs on this beat. This implies that no ball lands on beat i, so there is nothing available to throw. If $t_i \geq 1$, then the ball thrown on beat i will be in the air for t_i beats. Thus, this ball will land on beat $i + t_i$. In two hand juggling, at each beat we switch which hand is performing an action. Thus, if t_i is even, the ball will land in the same hand, and if t_i is odd, the ball will land in the opposite hand. There has been work on juggling with more than two hands. For an example of this, see [5], which explores two person juggling. In general, though, the number of hands will not affect the mathematics, as we will see later. Note that the throw height does not necessarily reflect the physical height of the ball. In physical juggling, two different jugglers may execute a 3 throw (throw height of three) to different physical heights, depending on their speed and style of juggling.

A *standard juggling sequence* (generally just called a juggling sequence) is a cyclic series of actions we perform on a collection of balls such that no more than one ball lands on a single beat. The number of actions in one such cycle is the *period*. For example, if we perform three distinct actions, any multiple of 3 would suffice as a period. The convention, though, is to use the smallest possible period.

The most common way to express a juggling sequence is with *siteswap notation*. The siteswap notation for a period n juggling sequence is an ordered sequence of the throw heights correlating to the action on each beat, $t_1, \ldots, t_n$. Often, since throw heights over 9 are uncommon, we drop the commas and simply write $t_1 \ldots t_n$. For example, we assume 102 means a period 3 juggling sequence with throw heights 1, 0, and then 2, and not a period 2 juggling sequence with throw heights 10 and 2.

Example 1. For our first example of a juggling sequence, we consider a period 2 sequence where the first throw height is 2 and the second throw height is 0. That is, on the first beat (which we will call beat 0), we throw a ball so that it will land

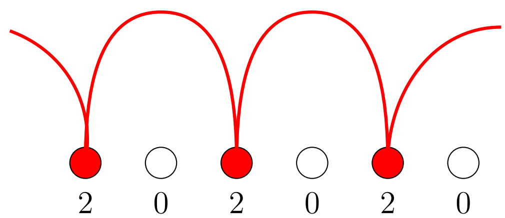


Figure 1: The juggling sequence 20

two beats in the future. Then, on the second beat (beat 1), we do not perform any action. By the third beat, the ball thrown on the first beat has landed. Since $3 \equiv 1 \pmod 2$, we again throw the ball two beats in the air, and on the next beat again perform no action.

Since we will throw the same ball for the entirety of this sequence, we say this is a 1-ball sequence. The sequence can be described by exactly two throw heights, so it is a period 2 sequence. In siteswap notation, this sequence is denoted 20. We could just as well write this sequencce as 2020 or 202020. However, we cannot write this sequence as 202, since this implies the sequence is period 3; then, the throw on beat 4 would be the same as the throw on beat 1, a 2-throw, which is not what we intended. Nonetheless, we generally write the sequence in the shortest manner possible, both for convenience, and in order to make the period clear.

See Figure 2.1 for a visual representation of this juggling sequence. Each circle symbolizes a beat, and we move left to right as time moves forward. Note that there is exactly one arc between any two beats, since we are only juggling one ball. We extend the path of our ball to the left and right of the picture to symbolize the fact that this is only a snapshot of the infinite sequence. Thus, while generally we consider the leftmost circle to correspond to time 0, the beginning/ending indices do not make a difference in our analysis.

Example 2. Another example of a juggling sequence in siteswap notation is 52512. See Figure 2 for the visual representation of such a sequence. From counting the number of distinct arcs in the air between any two consecutive beats, we see that such a sequence requires three balls.

As an aside, this sequence is dubbed the "baby pattern". In physical juggling, a 2 throw corresponds to holding the ball for two beats and a 1 throw is a horizontal toss from one hand to another. The "baby" in this sequence is the ball that travels via throws $2, 1, 2$ (one would hopefully shift, and not toss, the baby on the 1-throw). In Figure 2, this is the green ball.

Thus far, we have seen two different sequences of integers that each had a corresponding standard juggling sequence. However, because of our condition that only one ball can land on any single beat, there are many sequences of integers that cannot be juggled. Consider the following example.

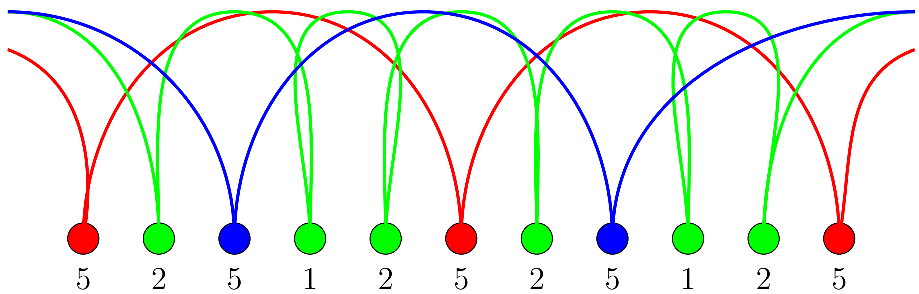


Figure 2: The juggling sequence 52512, also known as the baby pattern

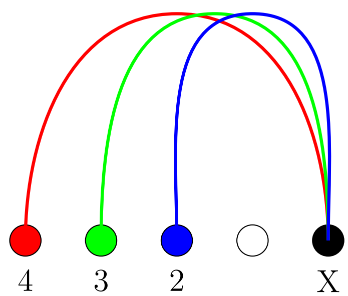


Figure 3: Attempting to juggle 432 will result in all three balls colliding at once

Example 3. Consider the sequence of integers $4, 3, 2$. If we attempted to juggle such a sequence, we would execute a 4-throw on beat 0, a 3-throw on beat 1, and a 2-throw on beat 2. Since none of the balls thrown will land while we carry out these three throws, we require three different balls. The first ball will land on beat $0+4 = 4$, the second will land on beat $1+3 = 4$, and the third will land on beat $2+2 = 4$. That is, all three balls will land on beat 4, which is not allowed in standard juggling. Thus, 432 is not a valid juggling sequence. See Figure 3 for an illustration of this event.

From Example 3, we see that distinct landing times are necessary for a sequence of integers to correspond to a juggling sequence. We provide the following definition.

Definition 4. A list of n nonnegative integers, $t_1 \dots t_n$, represents a *valid juggling sequence* if and only if, for all $1 \leq i \leq n$, the values of $t_i + i$ are distinct $\pmod{n}$.

This definition ensures that balls do not collide when landing. Since a ball thrown at time i lands t_i beats in the future, $t_i + i$ represents the landing time. We make sure that we do not have more than one ball landing at any time since we assume that we cannot catch more than one ball on one beat. Moreover, since juggling is periodic, we need only check the modular conditions. We can check that our juggling sequences in Example 1 and Example 2 are both valid. From the first example, we check that $2 + 0 \equiv 0 \pmod 2$ and $0 + 1 \equiv 1 \pmod 2$. Next, we can check that $5+0, 2+1, 5+2, 1+3, 2+4$ are all distinct modulo 5.

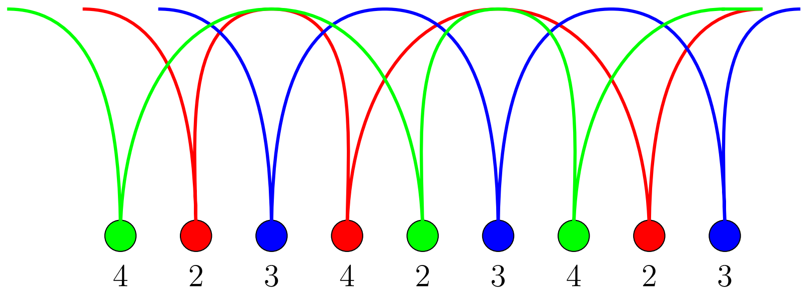


Figure 4: 423 is a valid juggling sequence since no more than one ball is caught/thrown on each beat

One can check that 25125 and 51252 are also valid juggling sequences. Since these sequences are meant to be infinite, it is not surprising that we can cyclicly shift the throws and still have a valid juggling sequence. We state this as a lemma.

Lemma 5. *All cyclic shifts of a valid juggling sequence are still valid juggling sequences.*

Proof. Let $t_1 \dots t_n$ be a valid juggling sequence, and suppose we want to shift by a, where $0 \le a \le n-1$. We have that $t_i + i \mod n$ are all distinct. Since there are n of these integers, $t_i + i \mod n$ is actually a permutation of $[n]$. Then, we have that $(t_i + i) + a \mod n$ is another permutation of $[n]$, so that there is still exactly one of each $0 \le j \le n-1$. □

From Lemma 5, we have a set of equivalence classes for each juggling sequence. There is no physical difference between juggling 52512 and 25125, so these sequences are essentially the same. Also, from earlier we have that 52512 and 5251252512 are the same sequence. There are infinite ways to describe a juggling sequence, and even if we limit ourselves to looking at sequences with the smallest period, there is still not a unique way to write a juggling sequence (unless it is period 1).

Recall that 3 balls are necessary to juggle 52512, which we determined by looking at an illustration. The following well-known theorem gives a quick way to determine the number of balls necessary from the siteswap.

Theorem 6. *[5] For a valid juggling sequence, $t_1 \dots t_n$, the number of balls necessary to juggle the sequence is exactly $\frac{1}{n}\sum_i t_i$.*

Proof. Let $t_1 \dots t_n$ be a period n juggling sequence with b balls. We sum the amount of time all the balls are in the air in two ways. First, since we immediately throw a ball after catching it, each of the b balls is in the air for n beats. However, since each t_i is the number of beats that the ball thrown at time i is in the air, we have that the collective air time is $\sum_i t_i$, including wraparound. This implies that

$$bn = \sum_{i=1}^{n} t_i$$

and the theorem immediately follows. □

A consequence of Theorem 6 is that the throw heights of a valid juggling sequence always average to an integer. The converse is not true, though. Recall in Example 3, that 432 was not a valid juggling sequence, even though the throw heights average to 3. However, if we rearrange the throws, one can check that 423 is valid. This is more than coincidence, as the following theorem shows.

Theorem 7 (Hall's Juggling Theorem [10])**.** *Let $s_1, \ldots, s_n$ be a sequence of nonnegative integers such that $\frac{1}{n}\sum_i s_i$ is an integer. Then, there exists some permutation π of n such that $s_{\pi(i)}, \ldots, s_{\pi(n)}$ is a valid juggling sequence.*

Hall's Theorem is sometimes called the partial converse of Theorem 6. For the traditional proof of Hall's Theorem, see [10]. Buhler and Graham also published a new proof in [6], which provides an algorithm for constructing the permutation π of $s_1, \ldots, s_n$ so that each $s_{\pi(i)} + i$ is distinct.

2.2 Bijection to Chessboards

The condition for a valid period n juggling sequence requires that there is some throw at each time $0 \leq i \leq n-1$ and that the values of $t_i + i \mod n$ are distinct. Moreover, there is a natural pairing between these sets; each i is associated with $t_i + i$. This connection between two permutations of $[n]$ is reminiscient of the conditions of a *non-attacking rook placement*. We provide a formal definition of this term

Definition 8. A *non-attacking rook placement* on an $n \times n$ board is a collection of rooks placed on the board so that there is no more than one rook in each row and column. That is, the collection of the cells occupied by rooks, $\{(i, j)\}$, are such that the i and j entries form two permutations of $[n]$. We refer to such a board as $\mathcal{B}_n$.

We next define an important class of juggling sequences, which will be useful to solidify the connection between rook placements and juggling sequences.

Definition 9. A *valid minimal juggling sequence* is a valid juggling sequence $t_1 \ldots t_n$ such that for all i, $t_i < n$.

Neither the baby pattern 52512 nor the pattern 423 is minimal. In order to make these minimal, we take each entry modulo the period. This gives the minimal counterparts 02012 and 120 respectively.

Observation 10. *Every valid juggling sequence can be uniquely mapped to a minimal valid juggling sequence.*

Minimal juggling sequences are the foundation of all juggling sequences. Given a valid juggling sequence, $s_1 \ldots s_n$, by reducing every $s_i \mod n$, we can create a minimal juggling sequence. We now illustrate the bijection between minimal juggling sequences and non-attacking rook placements.

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Figure 5: The rook placement that corresponds to the juggling sequence 33022.

Theorem 11. *There is a bijection between valid minimal juggling sequences of period n and non-attacking rook placements on an $n \times n$ chessboard.*

Proof. Let $t_1 \dots t_n$ be a valid, minimal, period n juggling sequence. On our chessboard, label the rows and columns $0, 1, 2, \dots, n-1$, starting in the left and on the top. For each $0 \leq i \leq n-1$, place a rook on the $n \times n$ grid, in row i and column $t_i + i \pmod{n}$. Since there is an action on every beat, there will be exactly one rook in each row. Moreover, by definition 4, all of the $t_i + i$ must be distinct $\pmod{n}$, so there will also be exactly one rook in each column.

In order to reverse the bijection, label cell (i, j) on the $n \times n$ board in the following way

$$\begin{cases} j - i & j \geq i \\ n + j - i & j < i \end{cases}$$

so that the label on (i, j) gives t_i in the corresponding minimal jugging sequence. Then, we can reconstruct our juggling sequence by reading the labels on the rooks, starting at the top row and working our way down. Note that each $t_i + i \mod n = j$, where j is the column index. Since there is one rook in each column, our $t_i + i$ are distinct, and the proposed juggling sequence is valid. □

Figure 5 shows the labeling of the board, which we will refer to as B_n, and a rook placement that corresponds to a valid minimal juggling sequence.

The bijection to rook placements makes the task of counting minimal juggling sequences of a fixed period easy. Since there are $n!$ nonattacking rook placements on an $n \times n$ board, so also there are at most $n!$ minimal juggling sequences of period n.

Unfortunately, this system overcounts in several ways. It will count every cyclic shift of a sequence separately. So, in reality, it is a better estimate to say there are $(n-1)!$ period n minimal juggling sequences. Also, approximating with chessboards counts sequences whose period divides the period. For example, the sequence 123 is a valid period 3, minimal juggling sequence. If we count that there are 6! period 6 minimal juggling sequences, we are counting the sequence 123123 and two cyclic shifts. We will also count sequences such as 345345, which can be reduced to a period 3 juggling sequence. However, 345 is not minimal. One could use an inclusion/exclusion argument to get an exact expression. For large n, though, the other terms we would be adding and subtracting would be small compared to $(n-1)!$. Thus, most mathematicians interested in juggling are content with approximating that there are $(n-1)!$ minimal juggling sequences with period n. For a more in-depth counting argument, see [3], which gives a formula for the number of juggling sequences of a fixed period and number of balls in terms of Möbius inversion.

Next, we analyze how the number of balls in a juggling sequence relates to the corresponding rook placement. Note that the juggling sequence 33022 requires two balls, and the corresponding rook placement in Figure 5 has exactly two rooks below the main diagonal (diagonal of all 0's). This equality will always be true, and is introduced as a lemma.

Lemma 12. *The number of balls in a minimal juggling sequence is equal to the number of rooks below the main diagonal in the corresponding rook placement.*

Proof. Consider a nonattacking rook placement on B_n with exactly b rooks below the main diagonal. Then, each of these b rooks is on a cell (i, j) with the label $n + j - i$, and the remaining $n - b$ rooks are each on a cell (i, j) with the label $j - i$. Since these labels give the throw heights of the corresponding juggling sequence, we can find the number of balls in this juggling sequence by Theorem 6:

$$\frac{1}{n}\sum_{i=1}^{n} t_i = \frac{1}{n}\left(bn + \sum_{j=1}^{n} j - \sum_{i=1}^{n} i\right) = b.$$

□

Lemma 12 makes the problem of counting juggling sequences with a certain number of balls more interesting since there are other, known methods that can be used to count non-attacking rook placements with a certain number of rooks below the main diagonal.

3 Eulerian Numbers

The Eulerian number, $\left\langle {n \atop k} \right\rangle$, was traditionally given as the number of permutations of n with k *ascents*, which is the same as the number of permutations of n with k *descents*. An ascent is a position i on a permutation $\pi = \pi_0, \pi_1, \ldots, \pi_{n-1}$ such that $\pi_i < \pi_{i+1}$, and a descent is the opposite, a position j in π such that $\pi_j > \pi_{j+1}$. For example, 1320 is a permutation of 4 with one ascent at position 0 and two descents at positions 1 and 2. So, if we use the convention of counting the number of descents, 1324 is counted by $\left\langle {4 \atop 1} \right\rangle$.

In [3], Buhler, Eisenbud, Graham, and Wright provide a bijection between permutations of n with k descents and permutations of n with k *drops*, also known as strict non-excedances. A drop is a position in a permutation π such that $i > \pi_i$. For example, 1320 has one drop at position 3. Hence, $\left\langle {n \atop k} \right\rangle$ also counts the permutations of n with k drops.

As in Theorem 11, we can create a bijection between permutations of n and rook placements on an $n \times n$ chessboard. If, for every $0 \le i \le n-1$, we place a rook on cell (i, π_i), we will have exactly one rook in each row and each column. Note that a drop in the permutation will result in a rook being placed below the main diagonal. Thus, we will use the term drop to refer to a rook placed below the main diagonal.

The connection between Eulerian numbers and rook placements gives us a new lens through which to study the properties of Eulerian numbers. We interpret two well-known properties of Eulerian numbers using rook placements.

Theorem 13. *The Eulerian numbers are symmetric; that is,*

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle.$$

Proof. Given a nonattacking rook placement with k rooks below the main diagonal, shift each rook one column to the right and count the number of rooks above the main diagonal. The rook previously in the last column is now in the first column and is either on or below the main diagonal. All k rooks below the main diagonal are either still below or on the diagonal. Then, the remaining $n-k-1$ rooks are above the main diagonal. If we swap the rows and columns of the board, we have $n-k-1$ rooks below the main diagonal. Since this process is invertible, we have a bijection. $\square$

Theorem 14. *The Eulerian numbers satisfy the following recurrence relation,*

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle.$$

Proof. We show that the number of $(n-1) \times (n-1)$ boards with $k-1$ drops times $n-k$ plus the number of $(n-1) \times (n-1)$ boards with k drops times $k+1$ is equal to the number of $n \times n$ boards with k drops. First, consider a rook placement on an $(n-1) \times (n-1)$ board with $k-1$ drops. Add a row to the bottom and a column to the right, and add a rook to the new cell (n, n). For any rook that is not on a drop, that is on a cell (i, j) such that $j \ge i$, switch the rows of this rook and the rook in (n, n). That is, we now have rooks on cells (i, n) and (n, j). This creates a new drop without removing any, so we now have k drops on an $n \times n$ board. Note that we have $(n-1)-(k-1) = n-k$ options for rooks to switch with. See Figure 6 for an illustration of this switch.

Next, consider a rook placement on an $n-1 \times n-1$ board with k drops, and again add a n-th row and column, and add a rook to cell (n, n). Now, select a rook that is on a drop, that is, a rook on a cell (i, j) such that $j < i$. Then, again, swap the rows of these two rooks. Note that this will not change the number of drops, so

Figure 6: One switch that builds a rook placement on an 5×5 board with 3 drops from a rook placement on an 4×4 board with 2 drops

Figure 7: One switch that builds a rook placement on an 5×5 board with 3 drops from a rook placement on an 4×4 board with 3 drops

we end with a rook placement with k drops on an $n \times n$ board. There are k rooks available for this, or we can also leave the rook on cell (n, n) and still have k drops. That is, there are $k+1$ options for rooks to switch (or not switch) with. See Figure 7 for an illustration of this switch.

To go backwards, given an $n \times n$ rook placement with k drops, if we find rooks on cells (a, n) and (n, b), with $a \neq n$ and $b \neq n$, swap the rows so that we now have rooks on (a, b) and (n, n), then delete row n and column n. Otherwise, if we already have a rook on cell (n, n), we do not have to switch anything, so we can immediately delete the last row and column without changing the number of drops. This operation is unique for any rook placement, since there is either exactly one pair $(a, n), (n, b)$, $a, b \neq n$ or a rook on (n, n), so the bijection holds. $\square$

This bijection allows us to make the following observation.

Observation 15. *The Eulerian number* $\left\langle {n \atop k} \right\rangle$ *counts the number of period* n*, minimal juggling sequences with exactly* k *balls.*

Armed with this observation, we seek a formula for the number of b ball, period n juggling sequences. Indeed, we have the following theorem, which gives a very concise formula for counting juggling sequences.

Theorem 16. *The number of period* n *juggling sequences with less than* b *balls is* b^n.

In order to reach this formula, though, we need to have a connection from any arbitrary juggling sequence to a minimal juggling sequence, since this is the category we can count the best. We begin by noting that we can easily add to the number of balls in a juggling sequence by changing the throw heights.

Lemma 17. *Given that $t_1 \dots t_n$ is a valid, k ball, juggling sequence, the sequence $s_1 \dots s_n$ where $s_i = t_i + b_i n$ for $b_i \in \mathbb{Z}$, and each $s_i \geq 0$, is also a valid juggling sequence, with $k + \sum b_i$ balls.*

Proof. Note that,

$$s_i + i \mod n = t_i + b_i n + i \mod n = t_i + i \mod n.$$

We also know that the collection of $\{t_i + i\} \mod n$ are all distinct by definition of a valid juggling sequence. Thus, $s_1 \dots s_n$ is also a valid juggling sequence.

We complete our proof by counting the number of balls in $s_1 \dots s_n$, given as

$$\frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n}(t_i + b_i n) = k + \sum_{i=1}^{n} b_i.$$

□

Given a period n, k ball pattern, $t_1 \dots t_n$, we can add n to $b - k$ terms in the juggling sequence, where we are allowed to repeat which terms we use, in order to build a valid b ball, period n juggling sequence. For example, from the 3 ball juggling sequence 423, we can build three 4 ball juggling sequences by adding 3 to exactly one of the throw heights: 723, 453, and 426. If we wanted to build a 5 ball juggling sequence, we would have to count the number of ways to add 3 twice into some combination of the three throw heights. There are 3 ways to add 6 to one of the throw heights, giving the sequences 10 2 3 (we include spaces between the throw heights here to prevent confusion), 483, and 429, and 3 ways to add 3 to two separate throw heights, resulting in the sequence 753, 726, and 456.

Now, we seek a more general way to count the number of ways to build a b ball, period n juggling sequence from a period n, k ball juggling sequence, given that $b \geq k$. In particular, we are counting the number of ways to add n to $b - k$ entries in the juggling sequence. Our problem becomes equivalent to partitioning $b - k$ terms into n distinct parts, where repetition and empty parts are allowed. We have the following well-known and useful lemma. A common stars-and-bars proof of this lemma is given in [3].

Lemma 18. *The total number of ways to add $b - k$ terms of n to a list of n throw heights is given by*

$$\binom{b-k+n-1}{n-1}$$

We wanted to use minimal juggling sequences initially simply because they have already been counted. However, minimal juggling sequences are also useful in that they cannot be built in this way from any other juggling sequence, since each throw height is less than n. We could generalize Lemma 17 to include subtracting from throw heights, but then we would have infinite options for juggling sequences with at least b balls. This leads to the following lemma, first given by Buhler and Graham in [?].

Lemma 19. *The total number of period n, b ball juggling sequences is*

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b-k+n-1}{n-1}$$

Proof. In order to count the total number of period n, b ball juggling sequences, we count the number of minimal juggling sequences with $0 \leq k \leq b$ balls, where each of these terms is given by $\left\langle {n \atop k} \right\rangle$. Note that the maxmium number of balls for a minimal, period n juggling sequence is $n-1$, which corresponds to the juggling sequence $(n-1)(n-1)\dots(n-1)$. Thus, we sum from 0 to $n-1$ Next, for each k, count the number of ways to build a b ball juggling sequence from each of the k ball juggling sequences, which utilizes Lemma 18. If $b < n-1$, then for all $k > b$, $\binom{b-k+n-1}{n-1} = 0$, and these terms do not contribute to the overall sum. □

We now have an expression for the total number of period n, b ball juggling sequences, but Theorem 16 has an even more concise expression. Before we can prove this, we need two more tools. The first is a well-known binomial identity, known as the Hockey Stick Identity, with several proofs given in [7],

Lemma 20.

$$\sum_{i=0}^{m} \binom{i+n-1}{n-1} = \binom{m+n}{n}.$$

The second tool is Worptisky's Identity.

Lemma 21. *[[15] Worpitsky's Identity]*

$$b^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}. \tag{1}$$

Now, we are ready to begin our proof of Theorem 16.

Proof. Given Lemma 19, we can build an expression for the number of period n juggling sequences with less than b balls.

$$\sum_{j=0}^{b-1} \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{j-k+n-1}{n-1}.$$

However, we can interchange the order of separation since there is no j in the first factor. This gives the equivalent expression

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \sum_{j=0}^{b-1} \binom{j-k+n-1}{n-1}. \tag{2}$$

We rearrange Expression (2) in order to use Lemma 20. For a given k, we have that our binomial runs from $\binom{n-k-1}{n-1}$ to $\binom{n+b-k-2}{n-1}$. However, for $0 \leq j < k$, $\binom{j-k+n-1}{n-1} = 0$, so we need only consider $j \geq k$. That is, the nonzero values of the binomial go from $\binom{n-1}{n-1}$ to $\binom{n+b-k-2}{n-1}$. Then, we can replace $j-k$ with i and have i run from 0 to $b-k-1$. This simplifies Expression (2):

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \sum_{i=0}^{b-k-1} \binom{i+n-1}{n-1},$$

and, by Lemma 20, we have that the number of period n juggling sequences with less than b balls is counted by

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{n+b-k-1}{n}. \tag{3}$$

By replacing k with $n-k-1$ in (3), and using Theorem 13, we manipulate (3) so that we can use Lemma 21:

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{n+b-k-1}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop n-k-1} \right\rangle \binom{b+k}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n} = b^n.$$

Thus there are b^n period n juggling sequences with less than b balls. $\square$

We can also use Theorem 16 to find the number of juggling sequences with exactly b balls.

Corollary 22. *The number of period n juggling sequences with exactly b balls is given by $(b+1)^n - b^n$.*

Proof. We have that there are $(b+1)^n$ period n juggling sequences with less than $b+1$ balls, and b^n juggling sequences with less than b balls. Thus, $(b+1)^n - b^n$ counts juggling sequences with exactly b balls. $\square$

4 Multiplex Juggling and Generalized Eulerian Numbers

Thus far, we have constrained our juggling sequences by allowing at most one ball to be caught and thrown on each beat. However, many jugglers will execute several throws at the same time. Practically, this makes it easier to increase the number of balls being juggled. We provide a formal definition of this category of juggling.

Definition 23. A *multiplex juggling sequence* with *hand capacity* c is a juggling sequence such that up to c balls can be caught/thrown on each beat. We express such a period n, hand capacity c juggling sequence as $T_1 \dots T_n$ where each $T_i = \{t_{i,1}, \dots, t_{i,c}\}$ is a (multi-)set of the throws on the i-th beat.

Since all of the throws on the i-th beat are instantaneous, the order of the $t_{i,j}$ at each i does not matter. Next, we specify which sequences of multi-sets are valid juggling sequences. In Definition 4, we make sure that no more than one ball lands at each time. Now, we must make sure that no more than c balls land at each time.

Definition 24. A *valid multiplex juggling sequence* of period n and hand capacity c is a list of (multi-)sets of integers, $T_1 \dots T_n$, such that each $0 \leq i \leq n-1$ appears exactly c times in $\{t_{i,j} + i \pmod n\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq c}}$.

Example 25. We consider two proposed hand capacity 2 multiplex juggling sequences, $[1,2],[2,2],[0,1],[2,3]$ and $[1,2],[2,2],[2,3][1,3]$. We consider the set $\{1+0, 2+0, 2+1, 2+1, 0+2, 1+2, 2+3, 3+3\} \mod 4 = \{1,2,3,3,2,3,1,2\}$. Since there are three 2's, three 3's, and no 0's, this fails to be a valid multiplex juggling sequence. For the second, we look at the set $\{1+0, 2+0, 2+1, 2+1, 2+2, 3+2, 1+3, 3+3\}$ mod $4 = \{1,2,3,3,0,1,0,2\}$. Since each integer $0,1,2,3$ appears in the set exactly twice, this is a valid hand capacity 2 juggling sequence.

The method for determining the number of balls in a multiplex juggling sequence is also similar to the method for standard juggling.

Lemma 26. *The number of balls necessary to juggle the valid hand capacity c, period n juggling sequence $T_1, \dots, T_n$ is given by*

$$\frac{\sum_{i=1}^{n} \sum_{j=1}^{c} t_{i,j}}{n}.$$

The proof of this lemma is omitted since it is identical to the proof of Theorem 6. Since the condition for a valid multiplex juggling sequence resembles the condition for a valid standard juggling sequence, the bijection between juggling sequences and rook placements also generalizes well to multiplex juggling. This bijection uses *c-rook placements*, rook placements with exactly c rooks in each row and column. Note that the condition for a minimal mulitplex juggling sequence is similar to that of standard juggling sequences. That is, a multiplex juggling sequence $T_1, \dots, T_n$ is minimal if all of the throw heights $t_{i,j}$ are less than n.

Theorem 27. *There is a bijection between valid minimal multiplex juggling sequences of period n and hand capacity c, and c-rook placements on an $n \times n$ chessboard.*

The proof for this theorem generalizes easily from Theorem 11, and thus will not be included.

Example 28. From Example 25, we have that $[1,2],[2,2],[0,1],[2,3]$ is not a valid juggling sequence, but $[1,2],[2,2],[2,3][1,3]$ is valid. By Theorem 27, we can also determine which of these sequences are valid by attempting to place both on $\mathcal{B}_4$.

Figure 8 illustrates that $[1,2],[2,2],[0,1],[2,3]$ is not a valid multiplex, hand capacity 2 juggling sequence because there are three rooks in both the third and fourth

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

Figure 8: Attempt to place $[1,2],[2,2],[0,1],[2,3]$ on $\mathcal{B}_4$

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

Figure 9: Attempt to place $[1,2],[2,2],[2,3],[1,3]$ on $\mathcal{B}_4$

columns, and none in the first column. If we attempted to juggle this sequence, we would be forced to catch/throw more balls than is allowed by our hand capacity.

However, $[1,2],[2,2],[2,3],[1,3]$ is valid, as is depicted in Figure 9. Note that, in a c-rook placement, we are allowed to have multiple rooks in the same cell. This simply corresponds to making the same throw multiple times on the same beat.

Note that the average of the throws of $[1,2],[2,2],[2,3],[1,3]$ is 4, and there are 4 rooks below the main diagonal in Figure 5. Indeed, we can generalize Lemma 12 to minimal multiplex juggling sequences.

Lemma 29. *The number of balls in a minimal, multiplex juggling sequence is equal to the number of rooks below the main diagonal in the corresponding rook placement.*

The proof of Lemma 29 is very similar to that of Lemma 12, so we omit it.

4.1 Generalized Eulerian Numbers

We wish to have an object that counts multiplex, minimal juggling sequences, similar to how the Eulerian numbers count standard, minimal juggling sequences. Thus, we generalize the Eulerian numbers to c-rook placements in order to count multiplex juggling sequences with hand capacity c.

Definition 30. The *generalized Eulerian number*, $\left\langle {n \atop k} \right\rangle_c$, counts the number of c-rook placements on an $n \times n$ board with exactly k rooks below the main diagonal.

Note that, if $c = 1$, the generalized Eulerian numbers are exactly the Eulerian numbers.

Moreover, by Theorem 27 and Lemma 12, which can be extended to c-rook placements, we have that $\left\langle {n \atop k} \right\rangle_c$ counts the number of period n, hand capacity c juggling sequences with k balls.

$\left\langle {n \atop k} \right\rangle_2$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
n=1	1								
n=2	1	1	1						
n=3	1	4	11	4	1				
n=4	1	11	72	114	72	11	1		
n=5	1	26	367	1492	2438	1492	367	26	1

$\left\langle {n \atop k} \right\rangle_3$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9
n=1	1									
n=2	1	1	1	1						
n=3	1	4	11	23	11	4	1			
n=4	1	11	72	325	595	595	325	72	11	1

Table 1: The first few rows of the $c = 2$ and $c = 3$ case of the generalized Eulerian numbers.

In terms of multi-set permutations, $\left\langle {n \atop k} \right\rangle_c$ counts the number of ways to distribute $\{1^c, 2^c, \ldots, n^c\}$ into n size-c multi-sets $S_1, \ldots, S_n$, where each $S_i = \{s_{i,1}, \ldots, s_{i,c}\}$, and, in the whole list, there are exactly k values $s_{i,j}$ such that $s_{i,j} < i$. This list of sets corresponds to the list of landing times for a juggling sequence. From Example 25, we calculated the set of landing times for a valid multiplex, period 4, hand capacity 2 juggling sequence $\{1, 2, 3, 3, 0, 1, 0, 2\}$. We split this into a sequence of 4 multi-sets, $\{1, 2\}, \{3, 3\}, \{0, 1\}, \{0, 2\}$. Note that, while we have listed the numbers in increasing order in each set, the order does not matter.

The k values of $s_{i,j} < i$ correspond to the drops in the landing times. Recall that, when place a juggling sequence on a chessboard, we place it in cell $(i, s_{i,j})$. Therefore, the rook corresponding to $s_{i,j}$, where $s_{i,j} < i$, will be below the main diagonal.

Table 1 gives the first five rows of data for the $c = 2$ case. These were generated with a Sage program that essentially counted each row with a brute force.

4.2 Symmetry

One of the first apparent features to the data in Table 1 is the symmetry across the rows. We note that the proof of Theorem 13, which gives the symmetry of the Eulerian numbers in the setting of a non-attacking rook placement, can be generalized to c-rook placements. This gives the following theorem.

Theorem 31. *There is a bijection between c-rook placements on an $n \times n$ board with k drops, and those with $c(n-1) - k$ drops.*

Chessboard proof. Consider a c-rook placements on an $n \times n$ board with exactly k drops. Move each rook one column to the right. Now, all c rooks in the last column are in the first column, so none of them is above the main diagonal. None of the k original drops can be above the main diagonal either. The remaining $c(n-1)-k$ rooks will now be above the main diagonal. Note that, besides in the last column, any rooks initially on the main diagonal are now above it. If we flip the board across the main diagonal, then, the resulting rook placement has exactly $c(n-1)-k$ drops. We can reverse the process to return to our original rook placement, so the bijection holds. $\square$

Symbolically, Theorem 31 gives

$$\left\langle {n \atop k} \right\rangle_c = \left\langle {n \atop c(n-1)-k} \right\rangle_c.$$

We can prove this theorem in terms of juggling sequences also, but first we need a few intermediate tools. We generalize these tools so that they apply to more sequences than simply those related to juggling.

Lemma 32. *If the sequence of (multi-)sets $T_1, \ldots, T_n$ satisfies the modular conditions for a multiplex juggling sequence, then for $b \in \mathbb{Z}$, the sequence $T_1+b, \ldots, T_n+b$ satisfies the conditions as well, where $T_i + b = \{t_{i,1} + b, \ldots, t_{i,c} + b\}$.*

Proof. We have that each $k \in [0, n-1]$ appears exactly c times in $\{t_{i,j} + i\}$, taking all terms modulo n. We also know that $k+b \mod n$ is a permutation of all $k \in [0, n-1]$. Thus, each k will still appear c times in $\{t_{i,j} + b + i\} \mod n$, since each element is simply shifted by $b \mod n$. $\square$

Lemma 33. *Suppose the sequence of (multi-)sets $T_1, \ldots, T_n$ satisfies the modular conditions for a juggling sequence, and let $\alpha \in \mathbb{Z}$ such that $\gcd(\alpha, n) = 1$, and let $\beta \equiv \alpha^{-1} \mod n$,then $\alpha T_{\beta 1}, \ldots, \alpha T_{\beta n}$ will also satisfy the modular conditions, where $\alpha T_{\beta i} = \{\alpha t_{\beta \cdot i,1}, \ldots, \alpha t_{\beta \cdot i,c}\}$.*

Proof. Since $\gcd(\beta, n) = 1$, $\beta i \mod n$ is a permutation of $[n]$, and we can reindex i by β so that each T_i goes to $T_{\beta i}$, taking the index modulo n. Then scale each $t_{i,j}$ by α.

$$\{\alpha t_{\beta i,j} + i\} = \{\alpha(t_{\beta i,j} + \alpha^{-1} i)\} = \{(\alpha(t_{\beta i,j} + \beta i)\}$$

We have that αi is also a permutation of $[n]$, so this new set of $\alpha t_{\beta i,j} + \beta i$ is simply a reordering of the original set and still satisfies the modular juggling condition. $\square$

We restate Theorem 31 in juggling language.

Theorem 19. There is a bijection between hand capacity c, period n minimal juggling sequences with k balls and those wth $c(n-1)-k$ balls.

Juggling Proof of Theorem 31. Let $T_1, \ldots, T_n$ be a k ball, hand capacity c minimal juggling sequence. From Lemma 33, we have that $-T_1, \ldots, -T_n$ satisfies the modular conditions of a juggling sequence since $-1 \equiv n-1 \mod n$ and $n-1$ is relatively prime to n. Note that this is not a valid juggling sequence since every $t_{i,j} \geq 0$, so $-t_{i,j} \leq 0$. Then, Lemma 32 gives that $n-1-T_1, \ldots, n-1-T_n$ also satisfies the modular conditions. Moreover, since all $t_{i,j} \leq n-1$, we have that the throw heights in each $n-1-T_i$ are between 0 and $n-1$, we have that this new list of sets is a valid minimal juggling sequence. We compute the number of balls in this new juggling sequence using Lemma 26,

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{c}(n-1-t_{i,j})}{n} = \frac{nc(n-1)-\sum_{i=1}^{n}\sum_{j=1}^{c}t_{i,j}}{n} = c(n-1)-k.$$

□

4.3 $\left\langle {n \atop k} \right\rangle_c$ for large c

Another feature of note in Table 1 is that the first three columns, corresponding to $k=0$,$k=1$, and $k=2$, are the same in both tables. That is, it appears $\langle {n \atop 0} \rangle_2 = \langle {n \atop 0} \rangle_3$, $\langle {n \atop 1} \rangle_2 = \langle {n \atop 1} \rangle_3$, and $\langle {n \atop 2} \rangle_2 = \langle {n \atop 2} \rangle_3$. Indeed, for a given n, k, there are only finitely many distinct values of $\left\langle {n \atop k} \right\rangle_c$. We can prove this using both rook placements and juggling sequences.

Lemma 34. *For all $c \geq k$,*

$$\left\langle {n \atop k} \right\rangle_c = \left\langle {n \atop k} \right\rangle_k$$

First, we restate Lemma 34 in the setting of a chessboard.

Lemma 35. *If a c-rook placement on an $n \times n$ board has k drops and $c \geq k$, then there are at least $c-k$ rooks on each cell along the diagonal.*

Chessboard Proof. We will prove Lemma 35 by induction on $c+k$. The base case is $k+c=1$. Since $c \geq k$, the only case where this is possible is a 1-rook placement with no drops. This corresponds to one rook on each entry on the main diagonal, and our claim holds.

Now, suppose that for all combinations c, k, such that $c+k < m$, the hypothesis holds. Let $c+k=m$, and let S be a c-rook placement on an $n \times n$ board with k drops. We can model a rook placement as a bipartite graph, with one set of vertices the rows, and the other set the columns. Then, there is an edge between two vertices, i and j, if and only if there is a rook on cell (i, j). A subset of rows of size t corresponds to ct rooks, which must be placed in at least t separate columns. That is, the neighborhood of such a subset of rows is at least size t. Thus, we can invoke Hall's Marriage Theorem, given in [9] and find a perfect matching in this graph. This corresponds to a 1 rook placement within S. Call this rook placement T, and suppose that T has s drops. Let S/T denote the rook placement with all the same rooks as in S except those also in T. Then, S/T is a $(c-1)$ rook placement with $k-s$ drops, we can use our induction hypothesis. That is, S/T has at least $(c-1)-(k-s)$ rooks on each cell along the diagonal. If $s \geq 1$, we are done. If $s = 0$, then T is the same rook placement as the base case, so that S still has at least $c - k$ rooks on each cell along the main diagonal. $\square$

Juggling Proof. If our hand capacity c is greater than the number of balls being juggled, k, then at each beat, we must have at least $c - k$ zero throws at each beat. There is a bijection between this sequence of sets of throws, and that from removing $c - k$ zeros in each set, obtaining a juggling sequence with hand capacity k. $\square$

Now we begin our task of counting our multiplex juggling sequences. We go through two cases, one where we ignore the number of balls and count $\langle {n \atop k} \rangle_c$ for all valid k. Then, we count two-ball, hand capacity 2 (or greater) juggling sequences for any n, and look at generalizing this process for any number of balls.

5 Counting for all k

We first take on the easier task of counting all hand capacity c, period n juggling sequences, ignoring the number of balls. Using Theorem 27 to form the bijection between these juggling sequences and c-rook placements on the $n \times n$ board, we also can associate each rook placement to an $n \times n$ matrix where $a_{i,j}$ is the number of rooks in position (i, j). Such a matrix will have nonnegative integer entries, and each row and column will sum to c. This is a convenient connection since matrices of this designation have already been counted. These values can be found in the OEIS: see A000681 for $c = 2$ [16], A001500 for $c = 3$ [17], and A172806 for $c = 4$ [18].

Even though these matrices have been counted, they do not seem very easy to work with. Motivated by entry A0000681, which gives the $c = 2$ case, we have the following bijection, which translates rook placements to a category of multi-set permutations. Note that Theorem 36 is valid for all c.

Theorem 36. *There is a bijection between c-rook placements on an $n \times n$ board and permutations of the multi-set $\{0^c, 1^c, \ldots, (n-1)^c\}$ such that the descent set of each permutation is a multiple of c.*

Proof. We again label the rows/columns from 0 to $n-1$ and collect the column indices of the rooks in the following way: starting in the first (top) row, read the column indices of each of the rooks, from left to right. We go down a row and repeat the process until we reach the bottom. Note that this guarantees we will have exactly c of each $0 \leq i \leq n - 1$. Also, if we treat our list of numbers as a permutation of $\{0^c, 2^c, \ldots, (n-1)^c\}$, we could only possibly have a descent when going from row i to row $i + 1$, since all of the column indices from the same row will be entered in ascending order. This forces our descent set to be comprised only of multiples of c, since each row has c rooks.

Next, consider a permutation of $\{0^c, 1^c, \ldots, (n-1)^c\}$ with the descent set consisting only of multiples of c. From this permutation, form a length n list of size c multi-sets, $T_1, \ldots, T_n$ by dividing the permutation after every c entries. Then, if each $T_i = \{t_{i,1}, \ldots, t_{i,c}\}$, place a rook in each cell $(i, t_{i,j})$. Since each T_i has c entries, there are c rooks in each row. Overall, there are exactly c of each $0 \leq i \leq n-1$, so there will be c rooks in each column. □

Corollary 37. *There is a bijection between hand capacity c, period n minimal juggling sequences and permutations of the multi-set $\{1^c, 2^c, \ldots, n^c\}$ such that the descent set of each permutation is either empty or consists of multiples of c.*

Proof. Since c-rook placements on an $n \times n$ board are in bijection with both of these combinatorial objects, they also in bijection with each other. □

Note in the above proof that the permutations of the multi-sets, divided into a list of length c multi-sets, correspond to the landing times of a juggling sequence.

The OEIS entry for number of size n square matrices with every row and column summing to 2, A0000681, denoted a_n, includes the recurrence relation

$$a_n = n^2 a_{n-1} - \frac{1}{2} n(n-1)^2 a_{n-2}.$$

This equation gives us a recurrence for hand capacity 2, period n juggling sequences. A final goal of this project is to find a recurrence for the generalized Eulerian numbers. Therefore, we investigate this recurrence in terms of rook placements, to see if it could be adjusted for rook placements with a fixed number of drops.

We illustrate this recurrence in the setting of permutations of $\{1^2, \ldots, n^2\}$ such that the descent set is comprised solely of even numbers (that is, we set $c = 2$.) For ease, we view these as permutations of n size-2 multi-sets $T_1, \ldots, T_n$ such that $t_{i,1} \leq t_{i,2}$. One example for the $n = 3$ case is $[1, 2], [1, 2], [3, 3]$. Let A_n be the set of these permutations.

The recurrence has two components. First, we count how many ways we can construct elements in A_n from elements in A_{n-1}. This construction includes several specific restrictions in order to prevent excessive overcounting. Even with this caution, there are some elements that will be overcounted in our process. We define and count this category of elements.

5.1 Construction

We begin with an example of building elements in A_3 from elements in A_2. Recall that an element in A_2 is a length 2 list of size two sets, with exactly two 1's and two 2's, and with the first entry in each set no greater than the second. One can check that the only elements in A_2 are $[1, 1][2, 2]$, $[1, 2][1, 2]$, and $[2, 2][1, 1]$. Suppose we have $[1, 1][2, 2]$. If we want to construct an element in A_3, we need to append a third multi-set, add two 3's somewhere in the list, and make certain that the entries in the sets still are in proper order. We will later cyclicly swap the sets, so we need a convention for the position of this new multi-set. We establish the convention to be the rightmost position. After adding this mutliset, place one 3 in the second position in this set. Then, either place the second 3 in the first position of the last set, or in the second position of any current sets. In the latter case, take the entry that was previously in this position, and put it in the first position of the last set. The first scenario gives the list $[1,1][2,2][3,3]$, and the second gives the lists $[1,1][2,3][2,3]$ and $[1,3][2,2][1,3]$. We can cyclicly shift each of these lists of sets (shifting the whole sets, not the individual elements) and still have an element in A_3. This is one place where we use caution to not construct too many things. If we allowed any permutation of the sets, then $[1,1],[3,3],[2,2]$ can be cosntructed from $[1,1][2,2]$. However, $[1,1][3,3][2,2]$ then can also be constructed from $[2,2][1,1]$.

We have that, from $[1,1][2,2]$, we can construct 9 elements in A_3. We can repeat this process for the other two elements in A_2 as well, in order to construct 27 possible elements of A_3. This count is still a little too large, but we will soon correct this.

First, we generalize our construction from A_{n-1} to A_n. For any element in A_{n-1}.

- Add n-th multi-set
- Place one n in the second position of n-th set
- Either place the other n in first position of n-th set or in second position of one of $(n-1)$ existing sets.
  - In latter case, take the entry that was previously in this position, and place it in the first slot of the n-th set.
- Cyclicly shift the list $n-1$ times.

Note that we have n options for placing the second n, and each constructed list has n cyclic shifts, counting a shift of 0. If we repeat this process for all elements in A_{n-1}, we construct $n^2 a_{n-1}$ elements of A_n.

We have that this will generate all elements of A_n. That is, given an element of A_n, either both n's are in the same set, or they are in different sets. If the n's are in the same set, simply remove this set. If each n is in a different sets, then our element is of the form $\ldots[b,n]\ldots[c,n]\ldots$. Suppose that $c \geq b$. Then, we can switch c with the n in the left set, and drop the set $[n,n]$. In either case, we produce a valid element of A_{n-1}.

5.2 Correcting Overcounting

Looking at a table, or counting with paper and pencil, one will find that $a_3 = 21$, even though we constructed 27 valid elements in the previous example. Thus, we must have constructed 6 of the elements of A_3 twice.

Consider $[1,1][2,3][2,3]$. We showed in the previous example that this can be constructed from $[1,1][2,2]$. However, from $[2,2][1,1]$, we can build $[2,3][1,1][2,3]$, and by shifting this cyclicly, we have a second construction of $[1,1][2,3][2,3]$. Similarly, $[1,3][2,2][1,3]$ can be constructed from either $[1,1][2,2]$ or $[2,2][1,1]$. Including 3 cyclic shifts of both elements, we have found the 6 elements that are constructed twice.

Note that the elements we overcounted of the form $\ldots[k,n]\ldots[k,n]\ldots$, where $1 \leq k \leq n-1$. In fact, elements of this type are exactly those that are constructed twice. Note that, when tracing an element of A_n of the form $\ldots[b,n]\ldots[c,n]\ldots$ back to an element of A_{n-1}, we swap c with n in the first set if c is at least as big as b. However, if $b=c$, then we could do the same with b and the n in the second set. Notice that if $b<c$, we can only trace the element back to $\ldots[b,c]\ldots$, since the ordered set $[c,b]$ is not allowed in the construction. The only other type of element constructed is one of the form $\ldots[n,n]\ldots$, and that is uniquely traced back to the element of A_{n-1} that consists of all other sets in the element. Therefore, it is exactly elements of the form $\ldots[k,n]\ldots[k,n]\ldots$ that are constructed from two distinct elements of A_{n-1}.

We begin by counting the number of ways we can have elements of A_n of the form $\ldots[k,n]\ldots[k,n]$ where the number of sets in between each $[k,n]$ is less than or equal to the number of sets before the first set $[k,n]$. That is, in A_4, we would count $[1,1][2,2][3,4][3,4]$, but not $[3,4][1,1][2,2][3,4]$, which is simply a shift away from the first. Once we have established this number, we can take cyclic shifts of each element in order to get the total number of overcounted elements.

There are $n-1$ options for k. The remaining $n-2$ sets in each element look like an element of A_{n-2}, with the actual numbers shifted. For example, in the set $[1,1][3,3][2,4][2,4]$, $[1,1][3,3]$ looks like $[1,1][2,2]$, an element of A_2. In general, the remaining sets in $\ldots[k,n]\ldots[k,n]$ look like an element of A_{n-2}, with $i \geq k$ shifted up by 1.

If n is odd, then we are considering sets of the form $\underbrace{---}_{a}[k,n]\underbrace{---}_{b}[k,n]$, where $a+b+2=n$, $a \geq 0$, $b \geq 0$, and $a \neq b$. There are $\frac{n-1}{2}$ options for b so that $b<a$. We can take n cyclic shifts of every such way to fill this list, for any b. Note that, after $b+1$ cyclic shifts, we have a list of the form $\underbrace{---}_{b}[k,n]\underbrace{---}_{a}[k,n]$. Since $a>b$, this is not of the form to be counted, so fortunately we are not overcounting elements in this category. Recall that the remaining $n-2$ sets have a one-to-one correspondence with an element in A_{n-2}. Therefore, there are $\frac{n(n-1)^2 a_{n-2}}{2}$ ways to have such an element of A_n in this case.

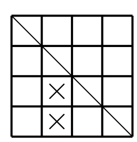
If n is even, then we must also consider the case $\underbrace{---}_{a}[k,n]\underbrace{---}_{b}[k,n]$ where $b=a$. For example, if we shift $[2,3][1,4][2,3][1,4]$ two times (shift each set two positions to the right cyclicly) we have an identical list. Including a shift of zero, we can only legitimately shift such a list $\frac{n}{2}$ times, when $a=\frac{n}{2}$. If $b<\frac{n}{2}$, we do not have this problem. Therefore, the number of valid cyclic shifts in this case are $\left(\frac{n}{2}-1\right)n+\frac{n}{2}=\frac{n^2}{2}-n+\frac{n}{2}=\frac{n(n-1)}{2}$. Therefore, we have that, in the even case as well as the odd case, our overcounted amount is $\frac{n(n-1)^2a_{n-2}}{2}$. This proves the second term of the recurrence.

The OEIS entries for $c=3$ and $c=4$ also have formulas and recurrences, but they include many terms, so we do not investigate these cases further.

6 Two ball juggling sequences

Now, we transition to include the number of balls in our counting of juggling sequences. For ease, we go in depth through a method for counting two ball juggling sequences, keeping in mind that, theoretically, we can do the same for any number of balls. Recall from Lemma 34, the only interesting hand capacities for a two ball juggling sequence are $c=1$, and $c=2$. Since the Eulerian numbers count the $c=1$ case, we solely consider $c=2$. We again use rook placements for the counting procedure. That is, we wish to count the number of ways to fill an $n \times n$ board with $2n$ rooks such that there are exactly two in each row and column and exactly two below the main diagonal.

We will illustrate the counting methods in this section by walking through the specific configuration of two drops in the same column. We begin with small example, shown below. We will carefully count the number of ways to fill this specific board, spending time to think about where and why we have more or less choices for positions to place rooks. Then, we will generalize this example so that we can count simultaneously the number of ways to fill many boards with two drops in the same column.



For this first example, we enumerate the number of ways to place the remaining six rooks on the board, on or above the main diagonal, as we move row by , from the bottom row to the top.

- In the bottom row, row 3, we only need to add one rook, and this can only be in the last column.

- In row 2, we only need to add one rook, and this can be in either the 4th or the 3rd column since neither is filled. That is, we have two options at this row.

- In row 1, we need to add two rooks. If previously, we had placed a rook in the 4th column, we can place both in the third column. Otherwise, if we had

Figure 10: The two possible ways to fill a board, given these two drops.

placed a rook in the 3rd column, we can now place one rook in the 3rd and one in the 4th. That is, we only have one possibility here, which will depend on the previous state.

- In row 0, one can check that, regardless of previous choices, all columns have exactly two rooks except the first. We place both rooks in this column.

There are two valid ways to fill this board while preserving the fact that this is a two ball juggling sequence. We display these in Figure 10.

In general, once we determine the positions of the drops, we enumerate the number of ways to position the remaining rooks on the board by placing rooks one at a time, beginning at the bottom row. Of course, our choices at later rows depends on those from earlier, but it will suffice simply to remember the state of the board in the previous row. The necessary information at each state is the number of columns that still need rooks, and how many rooks each needs. We write $\boxed{\ell}$ to denote a column that still needs ℓ rooks. Then, we define the *excess* at row m on an $n \times n$ board to be the total number of rooks that can be placed in columns $m, m+1, \ldots n$ *after* placing all necessary rooks in this row. One can think of the excess as the "leftover rook availability." The state of the row after placing rooks is simply a partition of the excess.

When we move from row m to row $m-1$, the excess can be changed in a couple ways. There can already be k rooks in the row $m-1$, below the main diagonal, so that we can only place $c-k$. This would increase the excess by k, since we place fewer rooks overall on this row. There can already be r rooks in the column $m-1$, so that this column has state $\boxed{c-r}$. Even though we are moving row by row, we call this case passing r rooks by column. This decreases the excess by r since our new column did not have full availability. Any row where the excess changes from the previous row is called a *transition row.* It is possible that the excess increases and decreases by the same amount, if there are k rooks in row $n-1$ and k rooks in column $n-1$. Such a row is still considered a transition row.

We review our previous example.

- We begin with 0 excess (denoted $\emptyset$). In row 3, we already have one rook to the left, so our excess increases to 1. Therefore, this is a transition row.

- In row 2, we have another rook to the left, and our excess increases to 2. This is another transition row.

- When we move to row 1, we have two rooks in column 1, so the excess decreases to zero again, and we have another transition row.
- Nothing of note occurs when we move to row 0, so the excess remains at 0.

Recall that, in the first and last rows, we have no choices in placing rooks. No matter what we choose in the middle rows, rooks in the first and last rows are always in the same location (see Figure 10). We have two options at row 2, when we transition from excess 1 to excess 2. The only possible state for excess 1 is $\boxed{1}$ since there is only one partition of 1. The possible states for excess 2 are $\boxed{2}$ and $\boxed{1}\boxed{1}$. That is, a row with excess 2 has, after placing rooks in the row, either one column that needs two more rooks or two columns that each need one more rook. We can count how many ways we can transition from excess 1 to excess 2. This transition requires that we move to a row that already has a rook. When we move to the next row, we add a new column that can accept up to two rooks. Since the previous row was at excess 1, there is also a column that still needs a rook to the right of the main diagonal. So, prior to placing rooks, we have a temporary state $\boxed{2}\boxed{1}$. We are only placing one rook, and this can be placed in either of these columns. If we place our rook in the column with availability 2, we will have state $\boxed{1}\boxed{1}$. Otherwise, we can place the rook in the column with availability 1, and end up with state $\boxed{2}$. The column we place the rook in will determine the final state. No matter where we place the rook, though, we end in excess 2. The following diagram illustrates this, with the starting state on top and the possible resulting states on the right. A 1 represents placing a rook in the column with the given excess, and 0 represents placing no rook in that column.

$$\begin{array}{cccc}
\boxed{2} & \boxed{1} & & \\
1 & 0 & \to & \boxed{1}\boxed{1} \\
0 & 1 & \to & \boxed{2}
\end{array}$$

We can express this information in a matrix where each column represents a starting state, each row represents a potential final state, and $a_{i,j}$ is the number of ways to start in state i and, after placing rooks, end in state j. The matrix for the transition from excess 1 to excess 2, then, is

$$\begin{array}{cc}
 & \boxed{1} \\
\begin{array}{c} \boxed{2} \\ \boxed{1}\boxed{1} \end{array} & \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{array}.$$

In the next row, the additional column we receive is already filled. There is only one way to place our two rooks, no matter the state. We form another transition matrix,

$$\begin{array}{cc}
 & \begin{array}{cc} \boxed{2} & \boxed{1}\boxed{1} \end{array} \\
\emptyset & \begin{pmatrix} 1 & 1 \end{pmatrix}
\end{array}.$$

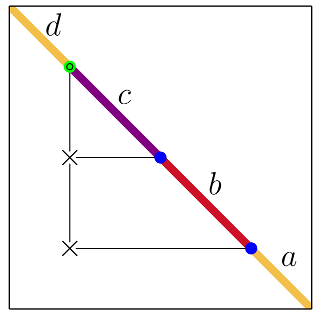


Figure 11: A generalized board with two drops in the same column. We will compute a generating function that gives the number of ways to fill this board with 2 rooks in each row and column and no additional drops.

Since the top row and bottom row have only one option for rook placement, their transition matrices are each simply (1). Then, we can count the total number of ways to fill this board by multiplying these transition matrices, from right to left. We show this below, with the row corresponding to each matrix labeled.

$$\underbrace{\begin{pmatrix}1\end{pmatrix}}_{0}\underbrace{\begin{pmatrix}1 & 1\end{pmatrix}}_{1}\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{2}\underbrace{\begin{pmatrix}1\end{pmatrix}}_{3} = 2.$$

We now wish to generalize this calculation to a rook placement with both drops in the same column, but with an arbitrary grid size n. We call this arbitrary board a *rook configuration.*

Definition 38. A *rook configuration* is a description of relative positionings of a collection of rooks on a chessboard of arbitrary size. For the purpose of this paper, the rooks we will position will be drops.

Figure 11 gives an illustration of this rook configuration. In this figure, a, b, c, d represent the rows between two transition rows, or between a transition row and an end of a board. In our example, $a, b, c = 0$ and $d = 1$. Rows in regions a and d have excess 0, rows in region b have excess 1, and rows in region c have excess 2. Note that this configuration requires a board that is at least size 3. It is impossible to have two drops in the same column but distinct cells in a 2×2 board. The fact that we require three rows is illustrated by the three lines from the drops in Figure 11.

So far we have mainly considered transitions between excesses of different amounts. However, we can do the same process for transitions between excesses of the same amount (this is still called a transition, even though there is no change in the excess). For example, consider a transition between two rows that each have excess 1. As our earlier illustration, when we get to the new row, there is a new available column to place rooks. Since there is no change in excess, there are no rooks in this new

Transition from

Transition to	$\emptyset$	$\boxed{1}$	$\boxed{2}$ $\boxed{1}\boxed{1}$
$\emptyset$	(1)	(1)	$\begin{pmatrix}1 & 1\end{pmatrix}$
$\boxed{1}$	(1)	(2)	$\begin{pmatrix}2 & 3\end{pmatrix}$
$\boxed{2}$ $\boxed{1}\boxed{1}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}$

Table 2: Transition matrices for excesses 0, 1, 2

row, and the new column we encounter has availability 2. Therefore, we have the temporary state $\boxed{2}\boxed{1}$, and we are placing 2 rooks. Our choices are illustrated below.

$$\begin{array}{cccc}
\boxed{2} & \boxed{1} & & \\
1 & 1 & \to & \boxed{1} \\
2 & 0 & \to & \boxed{1}
\end{array}$$

There are two ways to transition from excess 1 to excess 1, and there is only one state for excess 1, so this transition matrix is simply (2). The remaining transition matrices are in Table 2.

Then, for arbitrary values of a, b, c, d in Figure 11, we count the number of ways to fill the board by multiplying each type of transition the proper number of times, where each consecutive transition is multiplied on the left of the current term. This gives the expression

$$(1)^d \begin{pmatrix}1 & 1\end{pmatrix} \begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}^c \begin{pmatrix}1\\1\end{pmatrix} (2)^b(1)(1)^a.$$

Note that this expression is specifically for the case of two drops in the same column. If we have a different configuration, say two drops in the same cell, this expression will be different (although we would use the same table to set this up). For $n > 3$, there multiple values of a, b, c, d such that $a+b+c+d = n-3$, and we must account for all these combinations if we hope to count all rook placements for a fixed n. One way around this problem is to sum over every nonnegative a, b, c, d, and include a variable, x, in each factor in order to keep track of the number of rows used.

$$(x)^d \begin{pmatrix}x & x\end{pmatrix} \begin{pmatrix}2x & x\\x & 3x\end{pmatrix}^c \begin{pmatrix}x\\x\end{pmatrix} (2x)^b(x)(x)^a.$$

Next, we sum over all nonnegative a, b, c, d,

$$\sum_{a,b,c,d\geq 0} (x)^d x \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2x & x \\ x & 3x \end{pmatrix}^c x \begin{pmatrix} 1 \\ 1 \end{pmatrix} (2x)^b (x)(x)^a$$

In the matrices to a singular power, such as $\begin{pmatrix} x & x \end{pmatrix} = x \begin{pmatrix} 1 & 1 \end{pmatrix}$, we get a single power of x. This again keeps track of the fact that we need a board at least size three; even if $a = b = c = d = 0$, we have three singular power matrices, and we will get some constant times x^3. In other words, for nonnegative a, b, c, d, it is impossible to have an x or x^2 term from this expression.

Including our factor x in this way will result in the coefficient of x^n giving the number of 2-rook placements on an $n \times n$ board, with only two drops, which are in the same column. This is true since, for any combination of a, b, c, d, we could pull out the x terms to get $x^{a+b+c+d+3}$ times a product of matrices with integer entries. Adding like terms will get that the final coefficient of $x^{a+b+c+d+3} = x^n$ gives the total number of ways to fill an $n \times n$ board, for any valid $a + b + c + d + 3 = n$.

We simplify our expression by pulling out the single power x terms and distributing the sum to each relevant term.

$$x^3 \left(\sum_{d\geq 0} x^d \right) \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\sum_{c\geq 0} \begin{pmatrix} 2x & x \\ x & 3x \end{pmatrix}^c \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\sum_{b\geq 0} (2x)^b \right) \left(\sum_{a\geq 0} x^a \right) \tag{4}$$

We now have several geometric sums to compute. All are straightforward except the sum over the matrix. This sum can be computed in the same manner as other geometric sums. That is, for any square matrix M we have that $\sum_{n\geq 0} M^n = (I - M)^{-1}$. We carry out this computation,

$$\left(\sum_{c\geq 0} \begin{pmatrix} 2x & x \\ x & 3x \end{pmatrix}^c \right) = \begin{pmatrix} 1-2x & -x \\ -x & 1-3x \end{pmatrix}^{-1} = \frac{1}{1-5x+5x^2} \begin{pmatrix} 1-3x & x \\ x & 1-2x \end{pmatrix}.$$

Putting this in equation 4, as well as evaluating the other geometric sums, gives,

$$\begin{aligned} & x^3 \left(\frac{1}{1-x}\right) \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{1-5x+5x^2} \begin{pmatrix} 1-3x & x \\ x & 1-2x \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{1-2x} \frac{1}{1-x} \\ &= \frac{x^3}{(1-x)^2(1-2x)(1-5x+5x^2)} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1-3x & x \\ x & 1-2x \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{x^3(2-3x)}{(1-x)^2(1-2x)(1-5x+5x^2)}. \\ &= 2x^3 + 15x^4 + 75x^5 + 319x^6 + 1256x^7 + 4754x^8 + 17624x^9 + \cdots \end{aligned}$$

This generating function gives that, for example, there are 15 2-rook placements on a 4×4 grid with exactly two drops, which are in the same column.

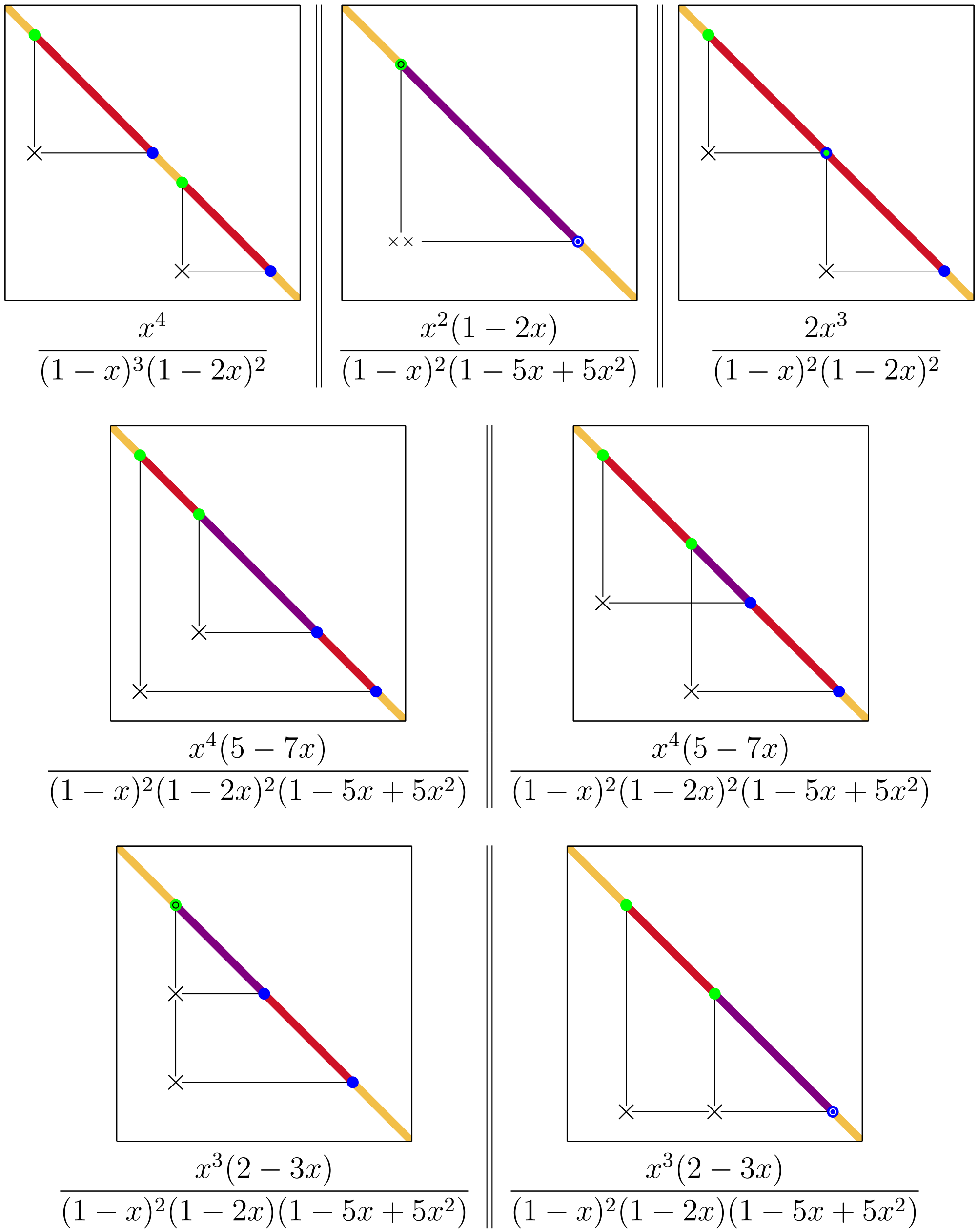


Figure 12: All generic 2-rook placements and corresponding generating functions.

We can repeat this process for every possible configuration of two drops in order to compute a generating function that can give the total number of two ball juggling sequences of any period n. The configurations and generating functions for the remaining cases are given in Figure 12

Note that, in the second and third rows of Figure 12, the generating functions are the same. The two configurations in the second row have the same sequence of row types (from excess 0 to excess 1 to excess 2, etcetera). In the third row, the row types are the same, only flipped in order. This provides a hint that the important part of a configuration of drops is simply the corresponding sequence of excesses of the rows. There will probably be more about this to come.

We sum all of these generating functions in order to have one function that gives the number of two ball juggling sequences for any n. This function is given as

$$\frac{x^2(-5x^4+3x^3+x^2+x-1)}{(1-2x)^2(x-1)^3(1-5x+5x^2)}$$
$$= x^2+11x^3+72x^4+367x^5+1630x^6+6680x^7+26082x^8+98870x^9+\cdots$$

Then, for example, there are 11 2-rook placements on a 3×3 board with exactly 2 drops.

7 Possible Recurrence

In [2], the author proves the unimodality of the Eulerian numbers using their recurrence. Since our generalized Eulerian numbers certainly seem to be unimodal, we first see if we can generalize this proof by finding a recurrence of these generalized Eulerain numbers. We will attempt to generalize the proof of Theorem 14 for a c-rook placement. For convenience, we will just examine the $c=2$ case. That is, given a $2(n-1)$-rook placement on an $(n-1)\times(n-1)$ board with k drops, we add an n-th column on the right and an n-th row on the bottom, and add 2 rooks in cell (n,n). Then, each of these rooks can either remain in this cell, or switch rows with one of the original $2(n-1)$ rooks. This can have several effects on the number of drops (and number of balls in the corresponding juggling sequence).

- If we keep the two new rooks in cell (n,n), or switch the rows of one or both of them with previous drops, then we will have a $2n$-rook placement with k drops. These are illustrated in Figure 13.

- If we switch exactly one of the new rooks with a rook from the original placement that is not a drop, we will have a $2n$-rook placement with $k+1$ drops. Note that the other new rook can either remain in cell (n,n) or switch with a previous drop. This is illustrated in the left board in Figure 14.

- If we switch both of the new rooks with rooks from the original placement that are not drops, we will have a $2n$-rook placement with $k+2$ drops. This is illustrated in the right board in Figure 14.

We prove the claim in the first bullet. The other items can be proved in a similar manner.

Naturally, if we keep both new rooks in cell (n,n), we have neither created nor destroyed any drops, so the final rook placement still has k drops. Suppose we switch

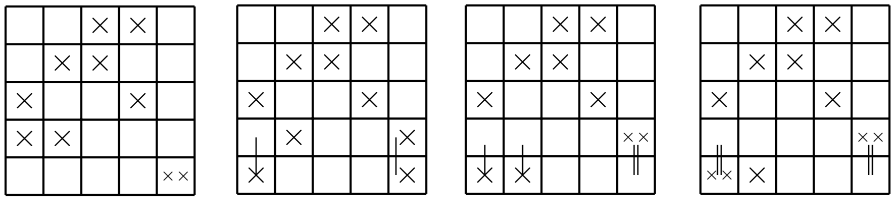


Figure 13: Four different ways to generate a 10 rook placement with 3 drops, given an 8 rook placement with 3 drops

one of the new rooks with a drop; that is, a rook in cell (c, d) where $n > c > d$. Then, we have rooks in cells (n, d) and (c, n). The rook in cell (n, d) is a drop, but the rook in cell (c, n) is not. That is, we destroyed one drop, and created a new drop. Similarly, if we do switch the second new rook with a second drop, we still preserve the number of rooks. In all of these cases, we end with a $2n$-rook placement with k drops.

Next, we will count the number of ways to achieve each type of row switching. In order to do this, we will establish two new variables: A, which gives the number of cells on and above the main diagonal containing rooks, and B, which gives the number of cells below the main diagonal containing rooks. Since we are restricted to allowing two rooks in each row and column, we have that $\lceil \frac{k}{2} \rceil \leq B \leq k$, where $B = k$ if each drop is in a separate cell and $B = \lceil \frac{k}{2} \rceil$ if as many drops as possible are in the same cell. Similarly, $\lceil \frac{2n-k}{2} \rceil \leq A \leq 2n - k$. Note that $k - B$ will give the number of cells below the main diagonal containing 2 rooks, and $(2n - k) - A$ will give the same number for cells on or above the main diagonal. Then, given a $2(n-1)$-rook placement on an $n - 1 \times n - 1$ board with k drops, we count the number of ways to create a $2n$-rook placement on an $n \times n$ board with k, $k + 1$, or $k + 2$ rooks.

If we want our new rook placement to have k rooks, we can either not swap any of the new rooks, or swap one or both with a previous drop.

- There is 1 way to not swap any rooks.
- There are B ways to choose a cell below the main diagonal with at least one rook, and swap that rook with one of new rooks.
- There are $\binom{B}{2}$ ways to select drops from two distinct cells to swap with the new rooks.
- There are $k - B$ ways to swap both of the new rooks with two drops in the same cell.

Then, in total there are $1 + B + (k - B) + \binom{B}{2} = 1 + k + \binom{B}{2}$ ways to create a $2n$-rook placement with k drops, given a $2(n-1)$-rook placement with k drops. See Figure 13 for examples of these four types of swaps.

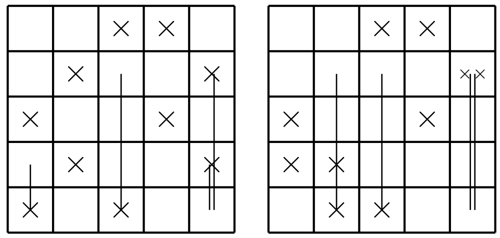


Figure 14: Examples of generating a new rook placement with 1(left) or 2(right) more drops than the original

If we want our new rook placement to have $k+1$ drops, we first need to select a rook on or above the main diagonal to swap with one of the new rooks. There are A choices for cells on or above the main diagonal with at least one rook that we can swap with one of the new rooks. For each of these A choices, there are $1+B$ choices for actions to make with the other new rook. That is, we can either leave that rook in cell (n,n), or swap it with one of the B cells containing rooks below the main diagonal. That is, there are $(1+B)A$ ways to create a $2n$-rook placement with $k+1$ drops, given a $2(n-1)$-rook placement with k drops.

If we want our new rook placement to have $k+2$ drops, we must swap both of the new rooks with rooks on or above the main diagonal. There are $\binom{A}{2}$ ways to select two rooks from distinct cells on or above the main diagonal. There are also $2n-k-A$ ways to swap both of the new rooks with two rooks in the same cell, on or above the main diagonal. Then there are in total $\binom{A}{2}+(2n-k)-A$ ways to construct a $2n$-rook placement $k+2$ drops, given a $2(n-1)$ - rook placement with k drops. Figure 14 has examples of generating rook placements with both $k+1$ and $k+2$ drops.

Note that this construction is different from that in section 5. Treat entry $t_{i,j}$ as a column index for a rook in row i, as described in Theorem 24. Recall that, in this construction for $c=2$, we add an n-th set, place one entry of n in this set, then place a second entry of n either in that set, or swap it with the larger entry in some other set, then put the displaced entry in the n-th set. In terms of a rook placement, adding an n-th set would add an n-th row and column. Placing one entry of n in this set places one rook in cell (n,n). If we place the second entry of n in the n-th set, this places the second new rook also in cell (n,n). Otherwise, if we place the second entry of n in the second position of set i, this places a rook in position (i,n). Then, if k was the integer in the second position in set i, we place k in set n, which places a rook in cell (n,k). Once we have placed the two new rooks, we will shift the board $n-1$ times cyclicly to the right.

While this construction successfully could build all of $2n$-rook placements from the $2(n-1)$-rook placements with an element of over-counting that was fairly easy to amend, it ignores any notion of rooks below the main diagonal. Thus, this construction is not useful for a recurrence that keeps track of the number of drops.

Like the first construction, though, this new construction also has an element of overcounting. Suppose we wish to find what $2(n-1)$-rook placement that mapped to a certain $2n$-rook placement. As for the $c = 1$ case described in Theorem 8, we look at the rooks in the n-th row and n-th column. Here, again, we have several cases.

- If there are two rooks in cell (n, n) already, simply delete row n and column n.
- If there is one rook in cell (n, n), and also rooks in cells (a, n) and (n, b) for $a < n$, $b < n$, we swap rows so that we have one rook in cell (a, b) and the other rook in cell (n, n). Then, we delete the last row and column.
- If there are no rooks in cell (n, n), but either in row n, column n, or both, there is a cell with two rooks, there is still one preimage. For example, suppose there are rooks in cells (a, n), (b, n), and two rooks in cell (n, c). Then, we swap rooks so that there are rooks in cells $(a, c), (b, c)$ and two rooks in (n, n), so that we can again delete the last row and column.
- If there are no rooks in cell (n, n), and the rooks in row n and column n are in distinct cells, then there are two preimages. That is, $2n$-rook placement with rooks in cells $(a, n), (b, n), (n, c), (n, d)$ where $a \neq b$, $c \neq d$, and a, b, c, d are all less than n, can be mapped back to a rook placement with rooks in cells $(a, c), (b, d)$ or rooks in cells $(a, d), (b, c)$. The left image in Figure 14 is one example of such a rook placement.

Let M_{2n} be the set of all $2n$-rook placements of the fourth form. That is, rook placements in set M are those that are constructed twice in our described construction. Also let $M_{2n,k}(A, B)$ be the number of $2n$-rook placements with k drops and B cells below and A cells on or above the main diagonal with at least one rook.

Then, we have the following sum for the number of $2n$-rook placements with k drops, building from $2(n-1)$-rook placements with $k, k-1$, or $k-2$ drops.

$$\left\langle {n \atop k} \right\rangle_2 = \sum_{A,B} \left[\left(M_{2(n-1),k}(A,B) \left(1 + k + \binom{B}{2}\right) \right) + M_{2(n-1),k-1}(A,B)((1+B)A) \right.$$
$$\left. + M_{2(n-1),k-2}(A,B) \left(\binom{A}{2} + (2n-k) - A \right) \right) \right] - M_{2n} \quad (5)$$

We investigate a method for counting the number of rook placements with certain values of A, B. When counting the number of rook placements for each possible value of B for a fixed value of k, we can use some of the same methods as discussed in section 6. For example, from Figure 12 we have that there are 6 possible cases for $k = 2$, $B = 2$ and one case for $k = 2$, $B = 1$. We determine all the cases for $k = 3$, $B = 2$ from those for $k = 2$, $B = 2$. We want to add a rook to one of the cells the already have a rook. Since we fix $c = 2$, we cannot have all three drops in the same row or column. There are 4 cases from $k = 2$, $B = 2$ where the original two rooks are in different rows and columns. For each case, we can add a rook to either of the cells that already have rooks, so that there are a total of 8 cases for $k = 2$, $B = 3$, although by symmetry some of the cases are essentially the same. These cases are drawn in Figure 15.

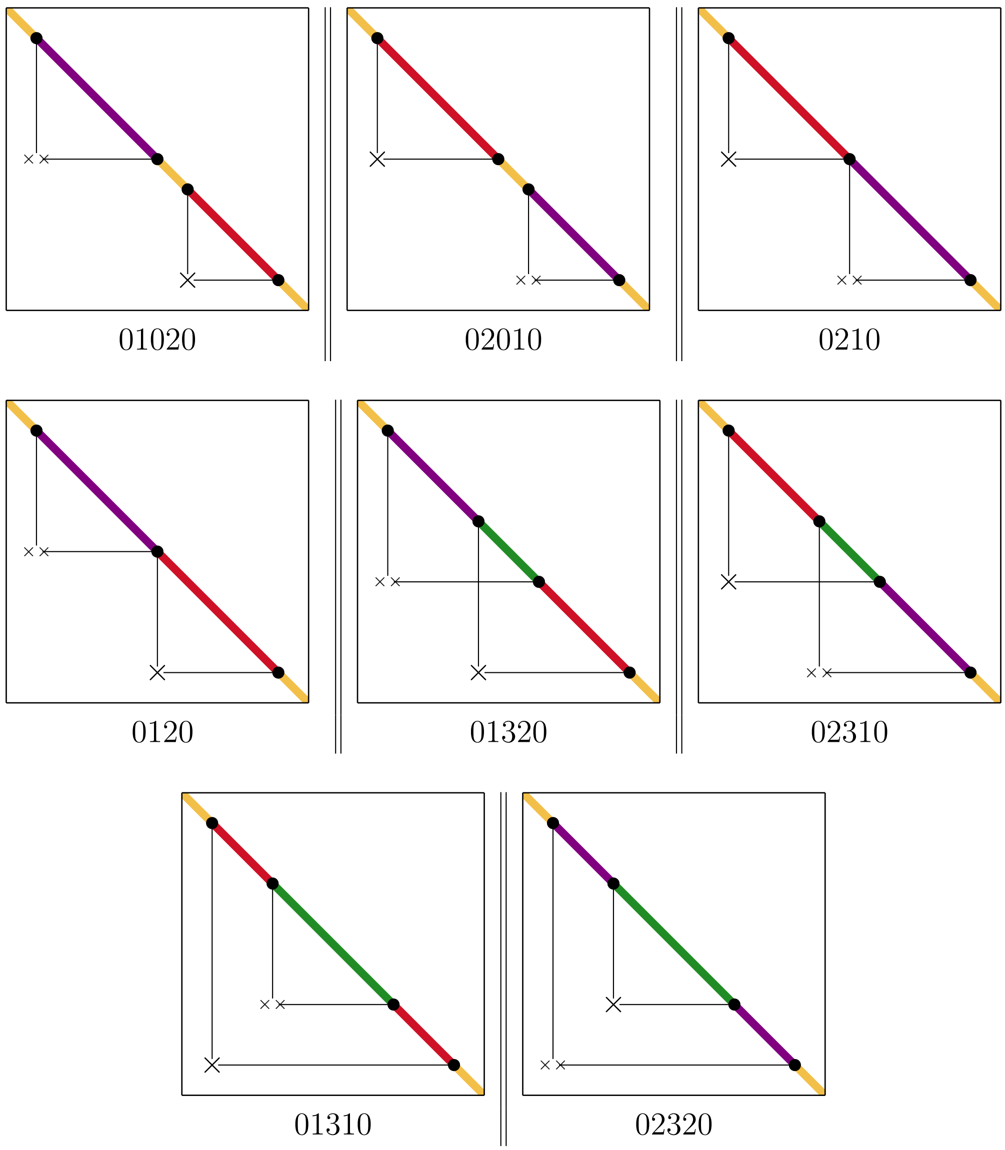


Figure 15: All generic $k = 3$, $B = 2$, with the associated excess words.

We could go through the same process as in Section 6 to find generating functions for each case to find the number of $2n$-rook placements corresponding to each drop configuration. However, we would require a new row and column in Table 2. Since we can partition 3 in three ways, these matrices will have up to three rows and columns. Then, for each new value of k, we would add another row and column to our transition

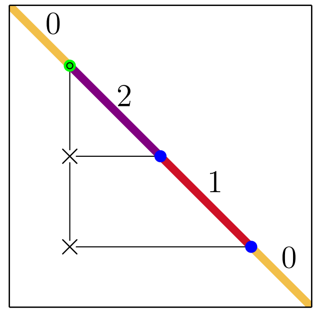


Figure 16: A generalized picture of a rook placement with two drops in the same column.

table, since we need a row and column for every possible value of the excess (which is the number of allowed drops k). Moreover, the new matrices we would form would grow even larger since the number of partitions of k grows quickly as k grows. It is also not clear if there is easy way to also keep track of the parameter A. Overall, this strategy does not seem particularly useful or practical for a full solution.

8 Excess Words

Since we have been interested several times in counting rook configuration (see Definition 38) with a certain number of drops, we make an attempt to count these by coding each with a word that describes the sequence of excesses encountered as we move up the board, row by row. Counting rook configurations could both perhaps make our counting via a generating function in Section 6 more efficient and help us write less vague formula for a recurrence than is given in Section 7.

Recall the analysis of the example rook configuration in Section 6, given again in Figure 16. As we work our way up the board, we pass through a section of excess 0, excess 1, excess 2, and then excess 0 again. That is, we can describe this whole board as 0120. We refer to this word that gives the consecutive regions of excess passed when moving row by row, from bottom to top, as an *excess word*.

In order to count these excess words, we need to clearly define what is and is not allowed. Recall our description of excess in section 6, as well as our notions of passing rooks by row or by column. For a fixed c, k, we have the following basic rules:

- The word starts and ends with a zero since we always start and end with excess 0.
- The excess increases by m if there are we pass m drops by row.
- The excess decreases by j we pass j rooks by column (that is, if we move to row r, there are j drops in column r, so that they are in this new column available

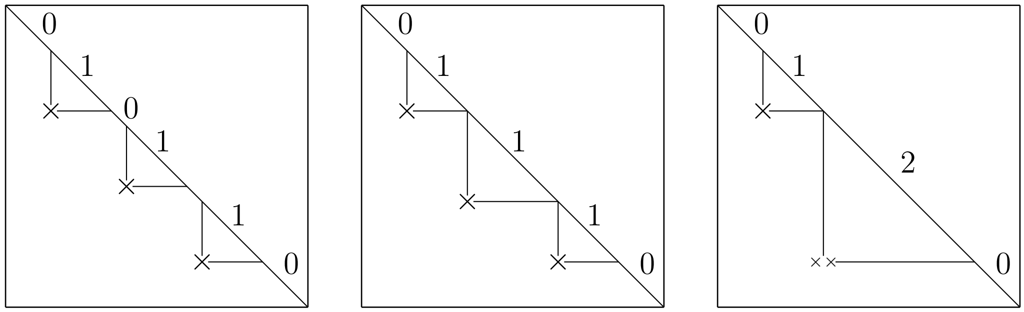


Figure 17: Examples for maximum and minimum length words for $k = 3$. The maximum length word is independent of c, but the minimum length word is shorter for larger c.

for placing rooks).

- It is possible for the previous two items to occur simultaneously, in which case our excess changes by $m - j$.
- For a fixed c, the difference between any two consecutive entries cannot be any greater than c, since the number of rooks we can encounter in a row or column can be no greater than c.

Note that a drop is in a cell (a, b) where $b < a$. Since we work from row $n - 1$ to row 0 for an $n \times n$ board, we must pass the drop by row before we pass it by column. This ensures that we will never have a negative excess since we will never pass more rooks by column than by row.

Note that every rook configuration has an infinite number of associated rook placements, since we can place as many rows and columns in between those with drops as we wish. However, since we require a certain number of distinct rows and columns for our drops that determine the word (as in our example in Section 6) we do have a minimum necessary size for each rook configuration. For example, the word 01110 requires at least 4 rows and columns, since we need three rows and columns below the main diagonal for our rooks. Generally, a word of length ℓ requires at least $\ell - 1$ rows and columns since we require $\ell - 1$ distinct transitions between excesses.

Table 3 and Table 4 give excess words for $c = 1$ and $c = 2$ respectively and $k = 1, 2, 3$. Even for these small values of c and k, several trends begin to appear. We prove several of these properities of excess words.

Lemma 39. *The length of an excess word is between* $\lfloor \frac{k}{c} \rfloor + 3$ *and* $2k + 1$.

Proof. In order to construct a minimal length word, we must maximize the number of rooks we meet at each step. For example, we start at excess 0, then we can pass up to c rooks in the first row with drops. Then, when we next encounter rooks, we can pass up to c column wise and c row wise. This does not change the excess at all,

	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = 7$
$k = 1$	010				
$k = 2$		0110	01010 01210		
$k = 3$			01110	011010 010110 012110 011210 012210	0101010 0101210 0121010 0121210 0123210

Table 3: Excess words for $c = 1$ and the first few values of k

	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = 7$
$k = 1$	010				
$k = 2$	020	0110 0120 0210	01010 01210		
$k = 3$		0210 0120	01110 02110 01210 01120 01320 02310	011010 010110 012110 011210 012210 012010 010120	0101010 0101210 0121010 0121210 123210

Table 4: Excess words for $c = 2$ and the first few values of k

so we still have excess c. We have thus far assigned $2c$ rooks to specific rows, and c rooks to a specific column. We continue until we have fewer than c rooks available to place in a row. If we have 0 rooks left, our word is of the form $0cc\dots cc0$, where we have $\frac{k}{c}$ entries of c. Otherwise, if r is the remainder of k divided by c, we can pass a row of r rooks, and a column of c rooks. This changes our excess by $r - c$. Since we have been at excess c, we are now at excess $c + (r - c) = r$. Then, in order to keep the word as short as possible, we next pass all of these r rooks in the same column. Now that all rooks have been passed both by row and by column, we are at excess 0. The resulting word here $0cc\dots ccr0$ with $\lfloor \frac{k}{c} \rfloor$ entries of c.

Conversely, if we want to maximize the length of a word, we do not want to pass rooks by both row and column at any step. In order to make the most transitions between regions of excess, we pass each rook individually by both row and column, then pass the next. This would generate a word of the form $0101\dots010$, where we have k entries of 1 for each of the k drops, and $k+1$ entries of 0. This is the maximum length of the word since each rook can, at most, contribute two changes to the word, an increase and a decrease of the excess. $\square$

See Figure 17 for examples of rook placements corresponding to maximum and minimum length words.

Lemma 40. *A number m can appear in a word for k drops at most $(k+1)-m$ times.*

Proof. For sake of contradiction, suppose that the number m appears $k-m+2$ times in a word. Note that the entry m means that, for some region of rows, there are m rooks we have passed by row but not by column. Moreover, we get a new entry if we pass some rook somehow, either by row or by column. Therefore, each of the $k - m + 2$ entries of m correspond to at least slightly different sets of rooks that have been passed by row and not by column. The best case would be that each of these sets have $m - 1$ of the same rooks, and 1 rook that is distinct for each. This requires that we have $m + (k - m + 1) = k + 1$ drops, which is a contradiction. Therefore, the entry m can only appear up to $(k + 1) - m$ times in a word. $\square$

Note that this makes sense even for $m = 0$. At most, we can use $k + 1$ zeros in between k entries of 1, as is the case for our example of a maximum length word. We also can never have two consecutive entries of zero, since this would require simultaneously passing a rook by row and by column, but there are no available rooks in a column below a region of excess zero.

Lemma 41. *If we add the amount increased at each ascent in a word, we get a number no greater than k.*

Proof. Every ascent in an excess word corresponds to passing more rooks by row than by column (we could perhaps pass zero by column also). Since we have k drops total, we cannot pass more than k rooks by row total. That is, the sum of the amounts increased at each ascent cannot be greater than k. $\square$

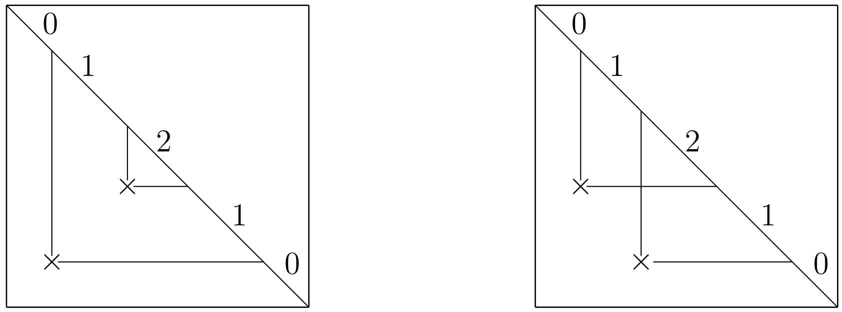


Figure 18: Nested and overlapping regions of excess can produce the same excess word.

For the $c = 1$ case, we have an even stronger claim.

Lemma 42. *Given a word for $c = 1$ and a fixed k and fixed length ℓ, the sum of the ascents in the word is $\ell - (k+1)$.*

Proof. Let a be the sum of the ascents in a word. Note that, since $c = 1$, the amount we can increase at each step is 1, so the sum of the ascents is the same as the number of ascents. Also, this is the same as the number of descents, since we start and end at 0. Let b be the number of positions where we neither ascend or descend. Then, we have that $2a + b + 1 = \ell$, where we add 1 because there are $\ell - 1$ positions in between the ℓ numbers in the word.

We have that each ascent accounts for one rook, since we pass one by a row. Also, every time two consecutive numbers in the word are the same, we pass a rook by row and another rook by column. That is, each ascent and each location with two consecutive numbers that are equal account for one rook, so that $a + b = k$. Combining this with our previous equation, $2a + b + 1 = \ell$, and solving for a, we have that $a = \ell - k - 1 = \ell - (k+1)$. □

Unfortunately, this rule does not apply for $c > 1$. For example, if we set $k = 4$, $c = 2$, $\ell = 5$, we have words 01220 and 01320, where our ascents add to two and three respectively.

One natural followup question could be how many specific rook configurations correspond to each excess word. Again, the answer is easier when we restrict to $c = 1$.

One of the main reasons we can have multiple configurations corresponding to each word is that two basic types of configurations, *overlapping* and *nested* rooks, yield the same word, 01210, as depicted in Figure 18. It follows that there are four ways to configure rooks to give 012101210 and 01211210, since we have two different locations where we have two choices for rook configurations.

Next, see Figure 19 for the $k = 3$ case. In order to have the word 0123210, we need to place rooks on the corners of the dashed lines. If we fix $c = 1$, then we

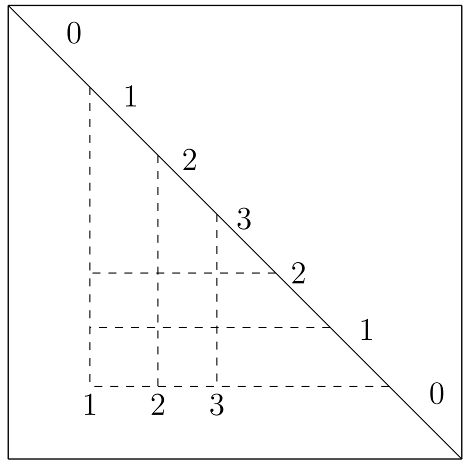


Figure 19: The dashed lines represent rows or columns that require a rook for this word, and the numbers under lines count the number of choices for placing a rook in that column.

essentially are placing a non-attacking rook placement on a 3×3 board, so that there are 3! ways to position the rooks. From here, we can deduce that the word $0123 \ldots k-1, k, k-1, \ldots, 3210$ has $k!$ options for corresponding rook configurations.

However, excess words are generally more complicated than these simple cases. We define a *peak* to be an ascent followed by a descent and conversely a *trough* to be a descent followed by an ascent. A word with a series of peaks and troughs would be of the form

$$0 \ldots p_1 \ldots t_1 \ldots p_2 \ldots t_2 \ldots p_k \ldots 0$$

, such as in the word 012321210. The next complication would be several consecutive numbers that are equal, such as in the word 012110. We provide the following lemma for counting the number of corresponding rook configurations for any excess word, with $c = 1$.

Lemma 43. *Given an excess word for $c = 1$, let $p_1, \ldots, p_k$ be the relative peaks in the word, and let $t_1, \ldots, t_k$ be the relative troughs in the word, with $t_k = 0$. Let $c_1, \ldots, c_m$ be the set of numbers in the word that appear consecutively, not necessarily pair-wise distinct, where c_i appears $n_i + 1$ times in a row. Then, the number of rook configurations corresponding to this word are*

$$\Pi_{j=1}^{k} \left(\frac{p_j!}{t_j!} \right) \Pi_{i=1}^{m} c_i^{n_i}$$

Before we justify this lemma, we demonstrate how to use this lemma. Given the word 01234322232110, we identify two peaks, 4 and 3, and the two corresponding troughs, 2 and 0. Also, we have that 2 appears 3 times consecutively, and 1 appears 2 times consecutively. Therefore, there are $\left(\frac{4!}{2!}\right)\left(\frac{3!}{0!}\right)(2^{3-2})(1^{2-1}) = 288$ corresponding rook configurations.

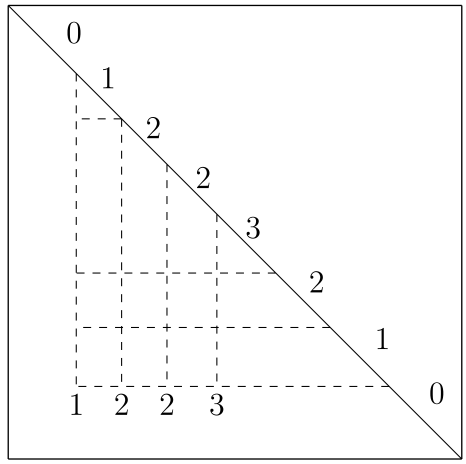


Figure 20: The dashed lines represent rows or columns that require a rook for this word, and the numbers under lines count the number of choices for placing a rook in that column.

To justify Lemma 43, we consider the structure of the rows and columns that must have a rook, which are determined by the excess word, like the grid in Figure 19. In order to satisfy the corresponding excess word, we place rooks on the intersections of the dashed lines of the grid so that, in the end, each row and column of dashed lines has a rook. In order to count the number of ways to do this, we go from right to left and count the number of ways to place rooks on the vertical lines. A vertical line corresponds to meeting a rook in a column. If we pass a rook by column while at excess m, there are m rooks that we have passed by row but not by column. In terms of our grid, there are m horizontal lines intersecting this vertical line that do not already have a rook, so that there are m options for where to place our rook on this vertical line.

We will have a vertical line if our excess decreases by one or if our excess remains the same. If our excess decreases by one and if there is another vertical line immediately to the left, there is one less option for horizontal lines to place a rook at this next line. If our excess remains the same, then at a vertical line immediately following this line, there is the same number of choices for positions to place a rook. The formula for our lemma follows from these facts. To see both of these principles, compare the choices in Figure 19 to those in Figure 20.

If we wanted to find a unique coding for a specific rook configuration, we could append to an excess word a permutation of the drops that gives the relative ordering of the drops. See Figure 21. Here, we pick as a convention that we list the relative heights of the rooks from left to right. In Figure 21, the relative ordering is 1342 because the leftmost rook is the highest on the board, the second rook from the left is the third highest, and so on. Note that several excess words can have the same relative ordering. For example, Figure 22 012321010 can also have ordering 1342.

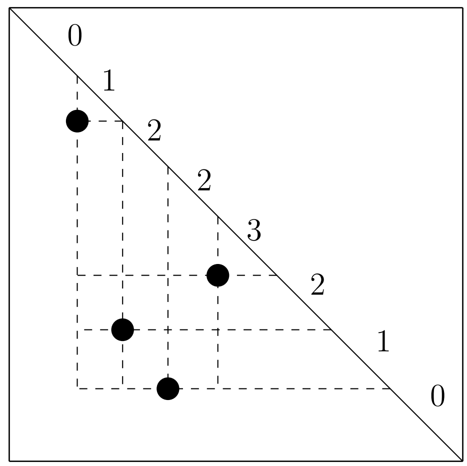


Figure 21: We can write this specific rook configuration as 01232210-1342.

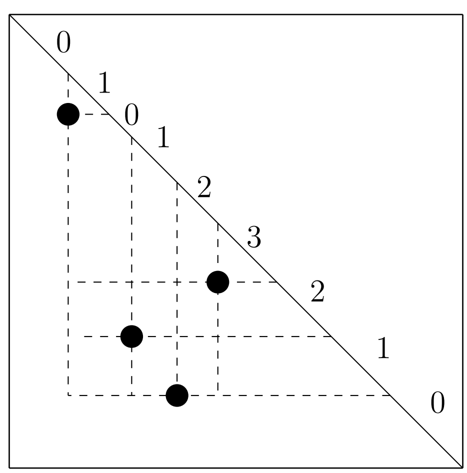


Figure 22: The word 012321010 can also have ordering 1342.

We lose the ease of counting for $c > 1$, however. For example, consider the word 01220 for $c = 2$. As depicted in Figure 23, this word can correspond to $k = 3$ and $k = 4$. The main issue with words for $c > 1$ is that it is not clear at each step how many rooks we are passing. For example, in 01220, when go between the two consecutive regions of excess 2, we can pass either one rook by both row and column, or two rooks by both row and column. The first step, going from 0 to 1, can only be done with one rook, since we have no rooks to pass by column also. At all other steps, we can pass either one or two rooks by row. The number of rooks we pass by column will depend on the required change of excess.

In general, our minimum number of drops for an excess word is given by the sum of the ascents plus the number of positions between two equal excess values. Therefore, 01220 requires at least 3 drops since our ascents sum to 2, and we have one position between two consecutive entries of 2. The maximum number of drops for an excess word of length ℓ is $m + c(\ell - 3)$ where m is the second entry (first nonzero entry). A length ℓ word has $\ell - 1$ transitions between regions of excess. At the first transition, we know we pass exactly m rooks by row; we cannot pass more by row because there are none to pass by column to compensate. Similarly, the last transition will only involve passing some rooks by column. If we pass rooks by row in the last transition, then we would not end at excess zero. At all remaining $\ell - 3$ transitions, we can pass c rooks by row. For example, 01220 can have no more than $1 + 2(2) = 5$ drops.

We have reached another point where the $c = 1$ case has nice results where $c > 1$ adds another complexity level. While excess words can be convenient for our process of forming generating functions, and have possible connections to other counting problems (for $c = 1$, the number of excess words for the first few values of k matched several sequences on the OEIS), it is not clear if they can provide any extra insight to the overall behavior of the generalized Eulerian numbers. We also reached a difficult complexity when trying to generalize the recurrence for the Eulerian numbers. In the end, I think more progress could be made in both of these problems, but more sophisticated counting methods are probably necessary. We leave these remaining problems as open questions.

- Is there a “nice” recurrence for the generlaized Eulerian numbers?
- Are the generalized Eulerian numbers unimodal?
- Is there a formula or algorithm for generating excess words for a certain c, k.

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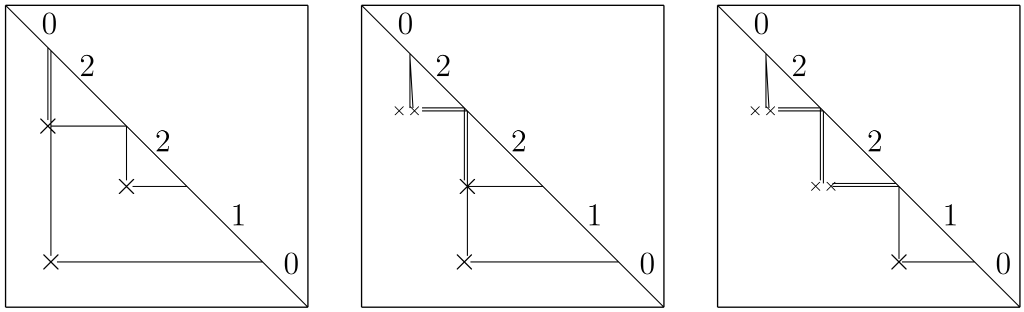

Figure 23: We can form the word 01220 for $c = 2$, $k = 3$(left), $k = 4$(middle), or $k = 5$(right).

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