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Bret Benesh

College of Saint Benedict/Saint John's University, bbenesh@csbsju.edu

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A Classification of Certain Maximal Subgroups of Alternating Groups

Bret Benesh

ABSTRACT. This paper addresses an extension of Problem 12.82 of the Kourovka notebook, which asks for all ordered pairs (n, m) such that the symmetric group S_n embeds in S_m as a maximal subgroup. Problem 12.82 was answered in a previous paper by the author and Benjamin Newton. In this paper, we will consider the extension problem where we allow either or both of the groups from the ordered pair to be an alternating group.

1. Introduction

While graduate students enrolled in a computational group theory course, the author and Benjamin Newton encountered problem 12.82 of the Kourovka Notebook [**5**]. This problem, submitted by V. I. Suschanskiĭ, poses the question of describing the set $\mathcal{M}$ of all pairs of positive integers (n, m) such that the symmetric group S_m contains a maximal subgroup isomorphic to S_n. One obvious family of such pairs is

$$\{(n, n+1) \mid n \geq 1\}.$$

The goal of the course was to provide an answer to this question with the help of the computational group theory system MAGMA [**1**]. A review of the literature indicated that a second family [**2**, **3**] was known:

$$\left\{ (n, m) \;\middle|\; m = \binom{n}{k},\; 2 \leq k \leq n/2 - 1,\; \binom{n-2}{k-1} \text{ is odd} \right\}.$$

MAGMA was used to check the maximal subgroups of symmetric groups of small degree, and it was determined that these two families did not constitute a complete solution to Suschanskiĭ's question. The data generated by MAGMA led to a discovery of a third family [**6**]:

$$\left\{ \left(kr, \frac{(kr)!}{(r!)^k k!} \right) \;\middle|\; k, r > 1,\; k + r \geq 6,\; (k, r) \in \mathcal{C} \right\},$$

where $\mathcal{C}$ is defined to be the set of all ordered pairs of the form $(2, 2^d + 1), (3, 2^e + 1)$, or $(2l, 2)$ for $d \geq 0, e \geq 1$, and $l \geq 2$. It was proved in [**6**] that these are the only three possible families, and all such ordered pairs lie in one of the three families. In

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this paper, we examine an extension of the question answered in the computational group theory course: the case when one or both of the groups in Suschanskiĭ's question is allowed to be an alternating group.

2. Preliminaries

We begin by stating the following three questions:

Q1: For what ordered pairs (n, m) does S_m have a maximal subgroup that is isomorphic to A_n?

Q2: For what ordered pairs (n, m) does A_m have a maximal subgroup that is isomorphic to S_n?

Q3: For what ordered pairs (n, m) does A_m have a maximal subgroup that is isomorphic to A_n?

We can answer the first question immediately with the following easy proposition.

PROPOSITION 2.1. *Suppose that a symmetric group S_n has a subgroup H that can be generated by a subset that only contains elements of odd order. Then H is a subgroup of the alternating group A_n.*   □

Since any group isomorphic to an alternating group can be generated by the images of 3-cycles, this proves that the only time S_m has a maximal subgroup isomorphic to A_n is if $n = m$.

To answer the remaining two questions, we simply need to look at maximal subgroups of the alternating group A_m. We will answer these questions by finding families of ordered pairs, and then showing that there can be no other ordered pairs outside of these families. There will be seven families that compose the answer to Q2, and these will be denoted $\mathcal{F}(S)_i$; the families that answer Q3 will be denoted $\mathcal{F}(A)_i$.

We reviewie a few basic facts about the maximal subgroups of symmetric and alternating groups. The following is well-known, and is not difficult to show.

PROPOSITION 2.2. *Let $m > 2$, X_m be either S_m or A_m, and M be a maximal subgroup of X_m. Then one of the following holds:*

(a) *M acts intransitively on $\{1, \ldots, m\}$ and $M \cong (S_k \times S_{m-k}) \cap X_m$, where $k \neq \frac{m}{2}$.*

(b) *M acts transitively but imprimitively on $\{1, \ldots, m\}$, $M \cong (S_r \wr S_k) \cap X_m$, where $kr = m$ and $k, r > 1$.*

(c) *M acts primitively on $\{1, \ldots, m\}$.*   □

The cases where the maximal subgroup does not act primitively are relatively easy and can be dealt with immediately. Suppose that A_m has a maximal subgroup M that is isomorphic to S_n, and that M acts intransitively on $\{1, \ldots, m\}$. Then M has the structure from Proposition 2.2(a), and it is an easy exercise to see that M must lie in the following family:

$$\mathcal{F}(S)_1 := \{(n, n + 2) \mid n \geq 3\}.$$

The only case where A_m has a maximal subgroup that is isomorphic to a symmetric group S_n that acts transitively but imprimitively on $\{1, \ldots, m\}$ is when $(n, m) = (4, 6)$. This ordered pair is already in $\mathcal{F}(S)_1$, although that instance represented an intransitive maximal subgroup isomorphic to S_4.

Now suppose that A_m has a maximal subgroup M that is isomorphic to A_n, and that M does not act transitively on $\{1, \ldots, m\}$. Then M has the structure from part (a) of Proposition 2.2, and it is again an easy exercise to see that M must lie in the following family:

$$\mathcal{F}(A)_1 := \{(n, n+1) \mid n \geq 3\}.$$

Finally, note that an alternating group A_n for $n \geq 5$ can never have the form of the wreath product from part (b) of Proposition 2.2, since the wreath product is not simple. For $n < 5$, we may check the cases individually to see that there are no maximal subgroups of the form described in part (b) that answer Q3.

3. The primitive case

For the remainder of the paper, we will be considering a subgroup X_n of A_m such that X_n is isomorphic to S_n or A_n, and that acts primitively on $\{1, \ldots, m\}$. Then X_n is in family (f) from [4] (all of the families (a) through (f) are listed after the following paragraph), and is therefore maximal in A_m unless one of following holds:

(1) $n = 6$ and $X_n < M \leq Aut(S_n)$, where M also embeds into A_m.
(2) $X_n \cong A_n$, and X_n is contained in the image of S_n in A_m.
(3) The pair (n, m) is explicitly listed as an exception in [4].

It remains to determine exactly when S_n and A_n act primitively on a set of cardinality $m \neq n$. To do this, we assume X_n acts primitively and we look at a point stabilizer H in X_n. Because the action of X_n on $\{1, \ldots, m\}$ is primitive, H is maximal in X_n, and $m = |X_n : H|$. The possibilities for H were enumerated in [4]:

(a) $H \cong (S_k \times S_{n-k}) \cap X_n$ where $k \neq \frac{n}{2}$ (the intransitive case).
(b) $H \cong (S_r \wr S_k) \cap X_n$ where $n = kr$ and $k, r > 1$ (the imprimitive case).
(c) $H \cong AGL(k, p) \cap X_n$ where $n = p^k$ and p prime (the affine case).
(d) $H \cong (T^k.(Out(T) \times S_k)) \cap X_n$ where T is a nonabelian simple group, $k \geq 2$, and $n = |T|^{k-1}$ (the diagonal case).
(e) $H \cong (S_r \wr S_k) \cap X_n$ where $n = r^k$, $r \geq 5$, and $k > 1$ (the wreath case).
(f) $T \lhd H \leq Aut(T)$ with T a nonabelian simple group, $T \neq A_n$, and H acts primitively on $\{1, \ldots, n\}$ (the almost simple case).

Moreover, [4] states that any subgroup of X_n of one of these forms is maximal, save for a list of explicit exceptions. The action of X_n on the cosets of a maximal subgroup yields a primitive action, and so we may simply consider the action of X_n on subgroups of the six forms listed above. We now only need to determine the values of n where S_n (respectively A_n) has a maximal subgroup of each type, taking into account the exceptions listed in [4]. Once again, MAGMA was useful in working with these exceptions.

Note that for cases (c)–(f), S_n embeds in A_m rather than the more general S_m by results from [6]. Therefore, if H is of one of these four types with $H \not\leq A_n$, then the image of A_n will always be contained in the image of S_n in A_m. In this case, the image of A_n will never be maximal. Similarly, we may conclude in cases (a) and (b) that A_m has no maximal subgroup isomorphic to A_n if the image of S_n embeds into A_n.

We will examine the six cases for H individually.

3.1. The intransitive case. Suppose now that H is intransitive. If $X_n \cong S_n$, then $H \cong S_k \times S_{n-k}$ $(k \neq n/2)$, and $m = |S_n : H| = \binom{n}{k}$. The results from [**2, 3**] and the exceptions from [**4**] tell us that we get the following maximal embeddings of S_n into A_m:

$$\mathcal{F}(S)_2 := \left\{ \left(n, \binom{n}{k} \right) \; \middle| \; \begin{array}{l} 2 \leq k \leq \dfrac{n}{2} - 1, \; \binom{n-2}{k-1} \text{ is even,} \\[2mm] \left(n, \binom{n}{k} \right) \neq (6,15), (10,120), (12,495) \end{array} \right\}.$$

If $X_n \cong A_n$, then $H \cong (S_k \times S_{n-k}) \cap A_n$ and $m = |A_n : H| = \binom{n}{k}$. This will not be maximal if the image of S_n is contained in A_m, which leaves:

$$\mathcal{F}(A)_2 := \left\{ \left(n, \binom{n}{k} \right) \; \middle| \; 2 \leq k \leq \dfrac{n}{2} - 1, \; \binom{n-2}{k-1} \text{ is odd} \right\}.$$

3.2. The imprimitive case. Suppose now that H is transitive but imprimitive. If $X_n \cong S_n$, then $H \cong S_r \wr S_k$ $(kr = n$, and $k, r > 1)$, and $m = |S_n : H| = \frac{(kr)!}{(r!)^k k!}$. The results from [**6**] and the exceptions from [**4**] tell us that we get the following maximal embeddings of S_n into A_m:

$$\mathcal{F}(S)_3 := \left\{ \left(kr, \dfrac{(kr)!}{(r!)^k k!} \right) \; \middle| \; k, r > 1, \; k + r \geq 6, \; (k,r) \notin \mathcal{C} \right\},$$

where $\mathcal{C}$ was defined in the introduction.

If $X_n \cong A_n$, then $H \cong (S_r \wr S_k) \cap A_n$ $(kr = n$, and $k, r > 1)$, and

$$m = |A_n : H| = \dfrac{(kr)!/2}{(r!)^k k!/2} = \dfrac{(kr)!}{(r!)^k k!}.$$

This will not be maximal if the image of S_n is contained in A_m, which leaves:

$$\mathcal{F}(A)_3 := \left\{ \left(kr, \dfrac{(kr)!}{(r!)^k k!} \right) \; \middle| \; k, r > 1, \; k + r \geq 6, \; (k,r) \in \mathcal{C} \right\}.$$

3.3. The affine case. Suppose now that $H \cong AGL(k,p)$ is affine; then n is equal to p^k. We will use the fact that the affine general linear group $AGL(k,p)$ is contained A_{p^k} iff $p = 2$.

If $X_n \cong S_{p^k}$, then S_{p^k} only has a maximal affine subgroup if p is odd. There is an infinite family of exceptions listed in [**4**] that occur when $k = 1$. This gives us an infinite family:

$$\mathcal{F}(S)_4 := \left\{ \left(p^k, |S_{p^k} : AGL(k,p)| \right) \; \middle| \; p \text{ is an odd prime, } k > 1 \right\}.$$

If $X_n \cong A_{p^k}$, then X_n will always be contained in the image of S_{p^k} unless $p = 2$. Then $X_n \cong A_{2^k}$ embeds maximally in A_m in the following conditions:

$$\mathcal{F}(A)_4 := \left\{ \left(2^k, |A_{2^k} : AGL(k,2)| \right) \; \middle| \; k \geq 3 \right\}.$$

3.4. The diagonal case. Suppose now that H is diagonal, let T be a non-abelian simple group, let $k \geq 2$, and let $D = T^k.(Out(T) \times S_k)$. Then $n = |T|^{k-1}$ and $H \cong D$. We would like to be able to determine exactly when this H is contained in A_n, but we only have an incomplete answer. It is known that H can lie outside of A_n iff one of the following occurs:

(1) $k = 2$ and $|\{t \in T \mid t^2 = 1\}| \equiv 2 \pmod 4$.

(2) $k > 2$, and $Out(T)$ contains an automorphism of T that acts of T^{k-1} via an odd permutation.

If $X_n \cong S_{|T|^{k-1}}$ and D contains an odd permutation of T^{k-1}, then $m = |S_n : D|$. This yields:

$$\mathcal{F}(S)_5 := \left\{ \left(|T|^{k-1}, |S_{|T|^{k-1}} : D| \right) \,\Big|\, k \geq 2, \, D \text{ contains an odd permutation} \right\}.$$

If $X_n \cong A_{|T|^{k-1}}$ and D contains only even permutations of T^{k-1}, then m is equal to $|A_n : D|$. This yields:

$$\mathcal{F}(A)_5 := \left\{ \left(|T|^{k-1}, |A_{|T|^{k-1}} : D| \right) \,\Big|\, k \geq 2, \, D \text{ contains only even permutations} \right\}.$$

There were no exceptions in [**4**] for this case.

3.5. The wreath case. Suppose now that $H \cong S_r \wr S_k$ for $n = r^k$, $r \geq 5$, $k > 1$. Then H always contains an odd permutation, and is never a subgroup of A_n; then the image of A_n will always be contained in the image of S_n and will never be maximal. We conclude that $m = |S_n : H| = \frac{(r^k)!}{(r!)^k k!}$, and we get the following two families of ordered pairs:

$$
\begin{aligned}
\mathcal{F}(S)_6 &:= \left\{ \left(r^k, \frac{(r^k)!}{(r!)^k k!} \right) \,\Big|\, r \geq 5, \, k > 1 \right\} \\
\mathcal{F}(A)_6 &:= \emptyset.
\end{aligned}
$$

There were no exceptions in [**4**] for this case.

3.6. The almost simple case. Suppose now that H is almost simple. We would like to determine when $H < A_n$, but this is currently an intractable problem. However, we can provide an implicit solution. Note that all simple groups are generated by their elements of odd order. Then by Proposition 2.1, all simple groups must be contained in alternating groups. Other almost simple groups, however, can lie outside of the alternating group.

We now consider the exceptions from [**4**] for the almost simple case. Define the four sets of exceptions as follows:

$$
\begin{aligned}
\mathcal{E}(S) &:= \left\{ (8, 120), (10, 2520), (22, |A_{24} : M_{24}|) \right\} \\
X(A) &:= \left\{ \begin{array}{l} (7, 15), (9, 120), (11, 2520), (23, |A_{24} : M_{24}|), \\ (175, |A_{176} : HS|), (275, |A_{276} : Co_3|) \end{array} \right\} \\
\mathcal{I}(A)_1 &:= \left\{ \left(c - 1, |A_c : \mathrm{Sp}(2d, 2)| \right) \,\Big|\, c = 2^{2d-1} \pm 2^{d+1}, \, d \geq 3 \right\} \\
\mathcal{I}(A)_2 &:= \left\{ \left(2^d - 1, |A_{2^d} : \mathrm{AGL}(d, 2)| \right) \,\Big|\, d \geq 3 \right\}.
\end{aligned}
$$

For convenience, define $\mathcal{E}(A) = X(A) \cup \mathcal{I}(A)_1 \cup \mathcal{I}(A)_2$. Then we get two more families of ordered pairs:

$$
\begin{aligned}
\mathcal{F}(S)_7 &:= \left\{ \left(n, |S_n : H| \right) \,\middle|\, \begin{array}{l} H \text{ almost simple, primitive,} \\ H \not\leq A_n, \, (n, |S_n : H|) \notin \mathcal{E}(S) \end{array} \right\}, \\
\mathcal{F}(A)_7 &:= \left\{ \left(n, |A_n : H| \right) \,\middle|\, \begin{array}{l} H \text{ almost simple, primitive,} \\ H \leq A_n, \, (n, |A_n : H|) \notin \mathcal{E}(A) \end{array} \right\}.
\end{aligned}
$$

4. Conclusion

Save for the implicit definitions the diagonal and almost simple cases, we may now explicitly answer the original three questions posed in this paper. These are complete lists of solutions to Q2 and Q3 because we have exhausted every possible type of maximal subgroup of symmetric groups and alternating groups by [**4**].

THEOREM 4.1. *The set of all ordered pairs (n, m) such that S_m has a maximal subgroup that is isomorphic to A_n is exactly*

$$\bigl\{(n, n) \ \bigm| \ n \geq 2\bigr\}.$$

THEOREM 4.2. *The set of all ordered pairs (n, m) such that A_m has a maximal subgroup that is isomorphic to S_n is exactly*

$$\bigcup_{i=1}^{7} \mathcal{F}(S)_i.$$

THEOREM 4.3. *The set of all ordered pairs (n, m) such that A_m has a maximal subgroup that is isomorphic to A_n is exactly*

$$\bigcup_{i=1}^{7} \mathcal{F}(A)_i.$$

5. Acknowledgements

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DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, ONE OXFORD STREET, CAMBRIDGE, MASSACHUSETTS 02138

E-mail address: benesh@math.harvard.edu