1995

Numeric Transmission/Cultural Transition

Paul B. Pekarek

College of Saint Benedict/Saint John's University

Follow this and additional works at: http://digitalcommons.csbsju.edu/honors_theses

Part of the Mathematics Commons

Recommended Citation

http://digitalcommons.csbsju.edu/honors_theses/531

Available by permission of the author. Reproduction or retransmission of this material in any form is prohibited without expressed written permission of the author.
Numeric Transmission / Cultural Transition

A THESIS

The Honors Program
College of St. Benedict / St. John's University

In Partial Fulfillment
of the Requirements for the Distinction “All College Honors”
and the Degree Bachelor of Arts
In the Department of History

by:
Paul B. Pekarek
May, 1995
Numeric Transmission / Cultural Transition

April 27, 1995

X Linda Lierheimer,
Assistant Professor of History.
Advisor.

X Jennifer Galovich,
Assistant Professor of Mathematics.
Advisor.

X Elaine Martin,
Associate Professor of English.
Reader.

X Martha Tomhave-Blauvelt,
Professor of History.
Chair, History Department.

X Margaret Cook,
Director, Honors Thesis Program.

X Tony Cunningham,
Director, Honors Program.
Table of Contents

Figure Table of Contents ................................................................. 6
Acknowledgments ............................................................................ 9
Introduction ....................................................................................... 11
A Brief History of the State of Science in the Classical, and Medieval periods .............................................................. 16
Popular Mathematics during the Middle Ages .................................... 24
Hindu-Arabic Numerals ..................................................................... 39
The dissemination of Hindu-Arabic numerals texts in Christian Europe ........................................................................ 43
The computational method associated with the Hindu-Arabic Numerals ........................................................................... 48
Manuscript Evidence ........................................................................ 50
Similarities Between the Abacus and the Algorist’s Method .................. 57
The Algorist / Abacist Controversy ...................................................... 63
Reasons for the transition from Roman Numerals to Hindu-Arabic Numerals ................................................................. 67
Conclusion ......................................................................................... 72
Figure Bibliography .......................................................................... 75
Annotated Sources ............................................................................ 78
References ......................................................................................... 94
Manuscripts ....................................................................................... 97
Catalogs ............................................................................................ 98
Appendices ....................................................................................... 100
Appendix A
  A Note on Cultural Relativism .......................................................... 101
Appendix B
  The Geometry of Gothic Church Windows ....................................... 103
Appendix C
  General Timeline ............................................................................ 104
Appendix D
  Glossary ........................................................................................ 106
Appendix E
Examples of Finger Counting Images.............................................107
Appendix F
Examples of Abacus........................................................................114
Appendix G
Explanation of Abacus Calculations.................................................116
Appendix H
History of Counting Board...............................................................118
Appendix I
Examples of the Reckoning Table....................................................121
Appendix J
An Explanation of Table Reckoning..................................................125
Appendix K
A Comparison Between Roman and Hindu-Arabic Numerals..........128
Appendix L
Examples of the Physical Transition that Occurred to the Hindu-
Arabic Numerals Over the Centuries................................................129
Appendix M
Al-Kwarismi Lexicon......................................................................131
Appendix N
A Brief Example of Algorist Multiplication......................................132
Appendix O
Examples of Algorist Division.........................................................133
Appendix P
Selection from Libro de moralidades, edited by George Greenia ....136
Appendix Q
Transcription and Translation of Wien MS#10585...........................140
Appendix R
Images of the Algorist/Abacist controversy....................................141
Appendix S
A Note on the Treviso Arithmetic and Similar Mathematical
Manuscripts.....................................................................................142
Appendix T
A Note on the Incorporation of the Hindu-Arabic Numerals in
Western Europe...............................................................................145
Appendix U
A Note on Society and Mathematics................................................146
<table>
<thead>
<tr>
<th></th>
<th>Figure Table of Contents</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Winged Arithmetic</td>
<td>10</td>
</tr>
<tr>
<td>2.</td>
<td>Aventinus on Finger Symbols</td>
<td>25</td>
</tr>
<tr>
<td>3.</td>
<td>Finger Counting Example</td>
<td>26</td>
</tr>
<tr>
<td>4.</td>
<td>Finger Symbolism in the 13th Century</td>
<td>28</td>
</tr>
<tr>
<td>5.</td>
<td>Roman Hand Abacus</td>
<td>30</td>
</tr>
<tr>
<td>6.</td>
<td>Roman calculator</td>
<td>31</td>
</tr>
<tr>
<td>7.</td>
<td>Abacus Multiplication Example 54x26</td>
<td>32</td>
</tr>
<tr>
<td>8.</td>
<td>Russian Abacus</td>
<td>34</td>
</tr>
<tr>
<td>9.</td>
<td>The Clerk and his Reckoning Table</td>
<td>35</td>
</tr>
<tr>
<td>10.</td>
<td>Apices Table</td>
<td>36</td>
</tr>
<tr>
<td>11.</td>
<td>Gerbertus’s <em>Apices</em></td>
<td>38</td>
</tr>
<tr>
<td>12.</td>
<td>Example of Early European Numerals</td>
<td>42</td>
</tr>
<tr>
<td>13.</td>
<td>The Salem manuscript</td>
<td>43</td>
</tr>
<tr>
<td>14.</td>
<td>Al-Khwarizmi’s Text in a Latin Translation</td>
<td>44</td>
</tr>
<tr>
<td>15.</td>
<td>The Carmen de Algorismo</td>
<td>46</td>
</tr>
<tr>
<td>16.</td>
<td>Algorist Method of Multiplication</td>
<td>48</td>
</tr>
<tr>
<td>17.</td>
<td>Wien #10585 Addition</td>
<td>55</td>
</tr>
<tr>
<td>18.</td>
<td>Wien #10585 Subtraction</td>
<td>55</td>
</tr>
<tr>
<td>19.</td>
<td>Wien #10585 Multiplication</td>
<td>55</td>
</tr>
<tr>
<td>20.</td>
<td>Wien #10585 Division</td>
<td>55</td>
</tr>
<tr>
<td>21.</td>
<td>Abacus Addition</td>
<td>58</td>
</tr>
<tr>
<td>22.</td>
<td>Abacus/Algorist Similarity</td>
<td>60</td>
</tr>
<tr>
<td>23.</td>
<td>Arithmetic’s Blessing of the Digits</td>
<td>63</td>
</tr>
<tr>
<td>24.</td>
<td>Computing Controversy</td>
<td>64</td>
</tr>
<tr>
<td>25.</td>
<td>The Merchant and his Counting Board</td>
<td>68</td>
</tr>
<tr>
<td>26.</td>
<td>The Son Apprenticed to the Abacist</td>
<td>69</td>
</tr>
<tr>
<td>27.</td>
<td>Geometry of Gothic Church Windows</td>
<td>102</td>
</tr>
<tr>
<td>28.</td>
<td>Boethius Finger-Counting</td>
<td>106</td>
</tr>
<tr>
<td>29.</td>
<td>Bede’s Finger Mathematics</td>
<td>107</td>
</tr>
<tr>
<td>30.</td>
<td>Finger Reckoning</td>
<td>108</td>
</tr>
<tr>
<td>31.</td>
<td>The Art of Finger Reckoning</td>
<td>109</td>
</tr>
<tr>
<td>32.</td>
<td>Finger Symbolism in the 13th Century</td>
<td>110</td>
</tr>
<tr>
<td>33.</td>
<td>Pacioli on Finger Symbolism</td>
<td>111</td>
</tr>
</tbody>
</table>
34. Chinese Abacus ................................................................. 113
35. The Salamis Abacus ......................................................... 113
36. Diagram of Abacus ........................................................... 116
37. Examples in the History of the Counting Board ..................... 117
38. Darius Vase ..................................................................... 118
39. The Treasurer on the Darius Vase ........................................ 119
40. Arithmetic Personified ....................................................... 120
41. Reckoners at Work ............................................................ 121
42. Reckoning Table Top .......................................................... 122
43. Reckoning Table ............................................................... 122
44. Discussion Over a Counting Table ...................................... 123
45. 1285 and 431 make 1716 ..................................................... 124
46. 284 times 153 is 43,462 ...................................................... 125
47. The Family Tree of the Indian Numerals ............................ 128
48. Table for the History of Hindu-Arabic Numerals .................. 129
49. Algorist Multiplication....................................................... 131
50. Galley Method of Algorist Division ..................................... 132
51. Galley Method of Algorist Division ..................................... 132
52. Strike-out Division ............................................................. 133
53. Strike-out Division in a Text ............................................... 133
54. Interpretive Galley Division ................................................ 134
55. Computational Dispute ...................................................... 140
Typographical and visuals assistance

by

Matthew J. Schneider
Acknowlegments

I would like to give special thanks to:

The Hill Monastic Manuscript Library, Fr. Eric Hollas, O.S.B. and staff for opening their doors to an undergraduate and being patient and helpful to see that I was accommodated.

Linda Lierheimer for her faith in my research and my ability to do well. Her enthusiasm was instrumental in being where I am today. Her advice was appropriate and helpful as many things changed these last months.

Jennifer Galovich for her guidance in my development as a scholar and as a man. Her careful expression of reality was a fantastic help for all areas of my life.

George Greenia for his positive cheering and faith in an inexperienced undergraduate. His reassurance carried me through many difficult times.

Elaine Martin for her interest and desire to live the “thesis experience” with me. Her patience was appreciated.

Sr. Wilma Fitzgerald for her sincere assistance in finding manuscripts and introducing me to a world of library resources while carefully showing me how to find my way.

Matthew J. Schneider for his unwavering support and love. The formatting he has done for my thesis and the work he has done scanning images is truly fantastic.

Mom, Dad and Dawn for their having stood behind me when I needed them.

All those who have been accommodating.
"Winged Arithmetic [is] shown here as the fourth of the seven liberal arts, in a woodcut by the Nuremberg artist Hans Sebald Beham (died 1550); she turns her back on the counting board (cloth) and points emphatically to the tablet with the new Indian numerals" (Menninger, 431).

A study of the computational methods and common use of mathematics during the Middle Ages and of the cultural transition that was associated with the numeric transition from the Roman numeral system to the Hindu-Arabic numeral system and the numeric systems' corresponding calculation methods in Christian Europe during the late Middle Ages and early Renaissance.
Introduction

This thesis examines popular Medieval mathematics and the transition that occurred when Hindu-Arabic numerals replaced Roman numerals in Western Europe in the late Middle Ages and Early Renaissance. The transition was not simply one in mathematics, but was related to social changes that sustained this transition in numeral systems. I will look at the social changes that occurred—especially the rise of mercantilism in the early Renaissance that propagated, encouraged and sustained this transition.

I claim that the transition from Roman numerals to Hindu-Arabic numerals was a result of cultural influences. Furthermore, I will show that it was a steady transition that found its basis in preexisting forms of calculation. The culture’s pre-existing mathematical use influences the culture’s receptivity to new methods. Mathematics influences the limits to the expression of a culture and therefore influences that society.

There is a cultural transition within economic and social spheres occurring during the later Middle Ages when Roman numerals are replaced by Hindu-Arabic numerals during the early Renaissance. This cultural transition I note is illustrated by Western Europe’s intellectual movement from its emphasis on allegorical meaning to an emphasis on pragmatic application fostered by the economic revitilization of the early Renaissance.

The numerals introduced to Europe had traveled from India to Arabia and then on to the Christian West; this is a trip which took over a millennium. The integration of these numerals into Europe is expressive of not only a cultural transition in the Christian West but of a dialogue between cultures. It is a cultural communication from the East to the West through numbers. It is a dialogue between the Arabic Islamic world and Western
Christianity, between the academic and lay communities; between the elite and the common people, between the merchant and the various communities with which he comes into contact, and between communities preferring the older methods and those preferring a different method. It is a discussion between elements within a society. Yet this dialogue of cultures does not always appear to be reciprocated.¹ Often, though, the cultures influence one another to move the entire community in a different direction.

I will explore the cultural and historical context of this transition from one numeral system to another by first discussing the states of society, science and math in the Middle Ages. I will then describe and compare the various computational methods that were available for use. I will discuss various computational methods and will note the similarities of the newer calculations using the Hindu-Arabic numerals to the pre-existing computational methods. I will then discuss some manuscripts to provide evidence to illuminate my claims about the popular comprehension of the newer methods of calculation and of the similarities between calculation methods.

Many of the secondary sources I studied perceived the state of mathematics during the Middle Ages as a “wasteland.” My study challenges this notion. A complete understanding of an element of society, mathematics for example, requires going beyond our modern bias of noting theoretical advances as comprising the entirety of culture to considering the culture that possessed and used certain methods on a practical level.² In this sense, there

¹For example, this study only notes the movement of Hindu-Arabic numerals from the East to the West. There was certainly a dialogue of cultures. Yet all aspects of such may not be presented in this thesis. There were other examples of influence travelling from West to East that are beyond the scope of this paper, such as the use of the abacus.

²In many of the secondary sources I consulted, I was struck by the bias the writers had for the elite “advancements” made in the pure and theoretical mathematics and against any pre-existing popular use of mathematical systems. Expressive of this is Hooper’s statement that: “From the days of Archimedes to the middle of the fifteenth century, very little real advance in
is a vast arena open to examine and understand the nature of the methods used by the populace. Consideration must be made for the role of the common person's comprehension of the elements that are used in his/her world. Although we may see little "advancement", this cannot prevent us from noticing, studying, understanding and appreciating a thriving system of mathematical method and comprehension. Frank J. Swetz in *Capitalism and Arithmetic*, notes how cultural relativism has not always been respected.

The early Renaissance has often been depicted as a rather stagnant period in the history of European mathematical development. Paul Rose, a mathematical historian, describes it as 'a large featureless plain broken up by such occasional peaks as Cardano, Copernicus and Galileo.' Still other authors tell us that no significant mathematics appeared in Europe between the time of the death of Fibonacci (1250) and the beginnings of the sixteenth century. Of course, such judgments are relative and rest on the interpretation of the work "significant." Significant for what and for who? If one equates significance with the appearance of original contributions in the form of theories that advance the corpus of mathematical knowledge, then the sentiments of Rose are correct. But there is more to mathematics than theory, for theory by itself, without a mode of expression or articulation, remains impotent. While not dramatic in the sense of new theories, the mathematical accomplishments taking place in the fourteenth and fifteenth period of transition for Western civilization, a movement toward new intellectual tradition, novel ideas and exotic practices imported from the East into Europe were taking hold. The conception of man and his place in society was changing. Man was no longer a passive observer. He became a 'doer' and a changer of his environment. It was for this end that Europeans, particularly the Italians, began to use mathematical knowledge took place"(84). An unparalleled example of the modern disregard for popular medieval mathematics is: "Beyond the cumbersome Roman numerals, which can be called a mathematical creation only by undiscriminating charity, the Romans created nothing even faintly resembling mathematics. They took what little they needed for war, surveying, and brute-force engineering from the Greeks they had rushed by weight of arms, and were content"(Bell 79). More indications of the modern perception of mathematics and civilization in general in the Middle Ages is Bell's chapter entitled: "The European Depression"(Bell xiii), which was a "... sterile period ... of the Dark Ages in Christian Europe"(Bell 79). Kline states that: "... There was nothing..."(92) and places his commentary on medieval mathematics in a chapter entitled "Interlude". "While European civilization rotted ... "(Bell 80), there was "intellectual stagnation ... "(Scott 53).
the Hindu-Arabic numerals and their associated algorithmic computational procedures (289-290).3

I will show that mathematics was used during the Middle Ages in various fields and the notion that little mathematics was used or understood because there were few individual contributing mathematicians during this period is simply an expression of a bias toward pure rather than applied science/mathematics. The notion that pure science or mathematics is better than its application is fundamental to believing that there was little advance in mathematics during the Middle Ages. For example, mathematics and geometry was used in the building of the cathedrals that stand as icons of the Middle Ages.4 Certainly different from what we would be willing to classify as scientific method or factual representation by modern standards, their practical method was strong and used abundantly.5 A statement that there was a waning of mathematical use denies the existence and popular usage of mathematical elements in the society.

Selected secondary sources played instrumental roles in the development of various components on this thesis. In the next section, I rely heavily on Edward Grant's Physical Science in the Middle Ages. This text displays some generals trends occurring within Western European academic and social spheres. Many of the images found throughout this thesis are due to the research of Karl Menninger's Number Words and Number Symbols. Although I question his conclusions because of his haphazard historiography,6 I found the examples of his evidence outstandingly comprehensive. Dr. George Greenia of the College of William and Mary was

---

3For other notes on the relative perceptions of medieval mathematics, see Appendix A.

4To note the use of geometry in Gothic church windows, I direct the reader to Appendix B.

5See the section on popular mathematics.

6Especially noted later will be the negative bias towards medieval mathematics or the lack of discussion of the popular element in mathematics that is presented in many secondary texts.
instrumental in collecting and helping with the manuscript sources and it is this information that forms the basis of the section dealing with manuscript evidence. In the later sections, the research of Frank J. Swetz in *Capitalism and Arithmetic* and *From Five Fingers to Infinity* will be noted extensively. His consideration of the societal impact on one element of social expression, the mathematics practiced by a culture, is illustrative of the concerns I wish to raise.
According to Edward Grant, in his work *Physical Science in the Middle Ages*, the works of Pythagoras, Plato and Aristotle stand as familiar landmarks to a great body of knowledge that the ancient world produced. Their ideas were revolutionary, inspired and creative, yet they were fundamentally theoretical. They hypothesized on the functioning of the universe and treated numbers with a special dignity above and beyond what is required in daily routine. Their works seem similar to what we would consider mathematics in schools and textbooks today. Yet, those who followed these ancient writers took a different developmental path.

For theory to exist in addition to what is practical, it needs to exist within a hospitable atmosphere to flourish. Edward Grant argues that a society must provide political stability, urban activity and patronage to some degree in order for the theoretical elements of science to flourish (2). When the societies that cultured this ancient growth began to disappear, then so did the luxury of expending time and energy in furthering theory. It is reasonable then, states Grant, to note that the deterioration of the theoretical elements in sciences had concrete societal foundations. However it would be biased to see this lack of advancement in the theoretical mathematics as an ending of the use of mathematics during the millennium following the fall of the Roman Empire. Given that mathematics is an integral part of society, we should recognize the importance of studying the common use of mathematics. It was

---

7In this section, I relied heavily on the work of Edward Grant to furnish appropriate information. I had read Grant carefully and have used this source heavily in the section on history since I found him to be the most objective and careful source. For an excellent summary of the individuals and texts that influenced academics and for the learning and development of mathematics for the educated from the ancient to the medieval period, see Grant, Ch. 1: “The State of Science from 500 A.D. to 1000 A.D.”, *Physical Science in the Middle Ages*, pp.1-12
8Appendix C contains various dates of prominent medieval mathematicians that may be helpful for the reader to place this discussion in the appropriate context of Western European history.
this popular use of mathematics that continued while the theoretical waned.

During the Hellenistic period, roughly 320-30 B.C., the ancient authors were revered and studied heavily and their influence reached across the Mediterranean basin. In the centuries that followed, what I have learned to be a typically Roman attribute came to rule over the sciences. Although the ancient authors were highly respected for their wisdom, they were at the same time considered impractical because their thoughts were often too lofty to find specific use or application. There was, in a sense, a body of academic mathematics dominated by pragmatism. In mathematics, as in other fields of study in the Roman world, usefulness governed what was taught. Although spanning various interests, the educators believed that only through application could academic study be justified. In this society, it was only necessary to complete the desired task. It was not important to understand the derivation of an idea or method nor was its context or relation to other areas considered.

Since education was not broad enough to handle the vast wealth of information of the ancient authors, a method was needed to distill the "necessary" information out of those works so that it could be applied in the Empire with greater ease. A new type of work was initiated that would be known as the encyclopedic/handbook tradition; a tradition which continued until after the fall of Roman Empire in the West (Grant 5). These works were a distillation of the ancient texts. In a way, they were translations not only from Greek to Latin, but were also from the language of the theoretical to the language of pragmatism. What happened then to the ancient authors was an evolution of the work. Authors such as Isidore of Seville (560-636) would pick and choose what would be needed to explain the process of an application with only preliminary evidence or foundations offered to the reader. Isidore
himself was atypical since he was very careful to document his sources while others felt no need to even explain the source or context of their information. Because context and explanation was not given to act as a foundation to understanding these works, the ancient works could not be understood any longer. During this period, a great deal of the theoretical works of the ancients was lost because it was simply beyond the comprehension of the next generation.

"Faced with little more than an unrelated collection of useless definitions, supplemented by a few trivial examples, the reader of Isidore's section on arithmetic could have put none of it to use."(Grant 11). For the academic world, this garbled mass is testament to the failure of the original pragmatic goal. For without understanding the intent of the work, how could one find its application? He who learned from this distilled body of knowledge, may then have attempted to teach without first having been able to understand the subject for himself. Because of a lack of documentation, an author or instructor was free to blatantly misinterpret a section of work. And since the ancient works were not fully understood before they were passed on, they were misunderstood even more the next time they were passed down. Since only the results and/or theories were disseminated and their context was not understood, the ancient works eventually were transformed into an unintelligible, garbled and incomprehensible body of work.

The expansion of Christianity had a profound influence on the history of mathematics. The first centuries of the Christian era saw a movement away from the hedonism of the Roman Empire. In this period, mystery religions and cults offered a way to express a common relinquishment of the temporal world and hope for a greater reward in the world beyond. A common thread that ran through them all was the belief that the world was
evil and would eventually pass away. Man, sinful by nature, could achieve immortal bliss if only he turned away from the things of this world and cultivated those of the eternal spiritual realm" (Grant 2).

Each of these cults vied for popular support. Christianity won the greatest support, and by 500 A.D. the Church was influential to the greatest minds of the age. As the Church gained support, all elements of life, including academia, were influenced. "Honor and glory were no longer found in objective, scientific comprehension of natural phenomena, but rather in furthering the aims of the universal Church" (Grant 4).

The academics of this time were thus able to wield control over the areas that were deemed important. "Pagan" studies, such as those that concerned themselves with attempting to understand the earth, were given little attention. Augustine is a fundamental example of the power that the Church and believers had over science. In his earlier years, Augustine was a proponent of mixing the traditional, pagan elements of the liberal arts into Christian learning. Yet, "a few years before his death, [St. Augustine] bitterly regretted his earlier emphasis on the liberal arts and concluded that the theoretical sciences and mechanical arts were in no way useful to a Christian" (Grant 5).9

9Swetz states in his introduction to the chapter "European Mathematics During the 'Dark Ages'" in From Five Fingers to Infinity that: "Pythagorean numerology and mysticism as evident in [texts of the first centuries A.D.] affirmed the Church's suspicion of the diabolical nature of mathematics as St. Augustine warned in A.D. 400, 'The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of hell'" (198) Dorothy V. Schrader states in "The Arithmetic of the Medieval Universities" (229-240) in Swetz, From Five Fingers to Infinity that St. Augustine was a defender of absolute truth whether it be found in pagan or Christian learning and that he specifically spoke of the unconditional truths of mathematics (230). A study of Augustine's On Christian Doctrine would be fundamental to understanding his opinions on mathematics. Schrader also mentions that the Church was helpful in educating people about mathematics by demanding its being taught. "In England, from the eighth to the twelfth centuries it was forbidden to the bishop to ordain a priest who could not compute the date of Easter and teach the method to
Considering the great influence of this Church Father, this
denunciation of mathematics likely held a great deal of weight.\textsuperscript{10} When
scientific treatises and other academic texts \textit{are} found during this period, they
often are an amalgam of fact, allegory and morality. It must be noted that the
influence of the Church was extraordinarily important in the academic realm.
Yet the Church played a lesser role in its influence on the common use of
mathematics.

The scientific texts of this era that we find bear little resemblance to
those to which we are accustomed today. During this period, a simple relaying
of facts was not deemed important; it was the metaphysical and spiritual
meaning that was fundamental. Facts without meaning were thought to be
useless.\textsuperscript{11} It is possible then that there may have been a great deal of work
that was lost because it did not fit entirely with the moral theme that was
desired. In this honing of information due to interpretive desires, combined
with the loss of work that was too advanced for comprehension, we no longer
see the ancient wisdom used extensively in the Latin West\textsuperscript{12}.

What knowledge of the ancient works remained was copied and
preserved in the monastic residences and schools for many centuries.
Cathedral schools began to teach this body of knowledge to the secular
community, which had begun to request education. During the ninth and
tenth centuries there was an expansion of education that was at least partially
due to the Carolingian renaissance. Education became available to the secular
community, though usually only to the elite. Perhaps this was because of the

\textsuperscript{10} One questions why society accepted the later declarations of this Church father while deciding
to disregard his earlier emphasis on integrating the pagan and Christian elements. Perhaps
people were simply more receptive to the removal of all pagan sciences from the Christian life.
\textsuperscript{11} The reader should not associate negative nor judgemental connotations to this interpretive
process. It is not "scientific" by our modern standards, though is no less a product of a society.
\textsuperscript{12} Later, I will discuss this concept in connection with finger-counting and medieval church
windows.
greater stability in the previously unstable world that followed in the centuries immediately after the fall of the Empire in the West. Because men now were free from threats to their survival, they were able to take advantage of the opportunity for education in the ninth and tenth centuries.

Gerbert of Aurillac (ca.946-1003), (also referred to as Grebertus, Pope Sylvester II (999-1003)) began an effort toward incorporating ancient works into Western education. Seeing that there was little left of the ancient works in science (since what came to him were the confused versions left from the handbook and the encyclopedic traditions), Gerbertus used his Church influence to obtain manuscripts from northern Spain. Charles Linn, in Mathematics East and West, notes that Spain had been a cultural center long before the twelfth and thirteenth centuries. During the period following the fall of the Roman Empire when the Latin West was reinterpreting the ancient works, yet losing the fundamental properties of those works, the Arabic empire had kept alive the writings of the ancient authors in translation. Many of these works resided in vast libraries in Moorish (Islamic) Spain. Gerbertus took these works, which were then translated into Latin, and used them to teach his students. Inspired by their teacher’s pedagogic ingenuity when emphasizing this new study as an integral part of the liberal arts, those students went out with enthusiasm to bring this “new” information across Europe (Grant, 14). His followers revived learning in the eleventh and twelfth centuries by integrating Gerbertus’ emphasis into the educational programs of the Cathedral school (Grant 14).

What began after this injection of ancient thoughts and scientific ideas across Europe was an amazing rush for knowledge by those who recognized their limitations regarding knowledge from the ancient sources. "The increasing concern for intellectual pursuits brought with it a greater interest
in the works of antiquity" (Grant, 15). This renewed enthusiasm for learning and the ancient works necessitated a great push for translations. Since the Arabic translations were the storehouse of the ancient authors, it was important for Arabic texts to translated into Latin. Grant notes that: "Between 1125 and 1200, a veritable flood of translations rendered into Latin a significant part of Greek and Arabic science" (Grant, 15).

During this period, Christians were physically removing Moslems from European land. Although the Moslems were pushed out, their culture remained in the libraries which were left standing. “A now dynamic Christian Europe came into possession of great centers of Arab learning” (Grant 16). Europeans were eager to utilize this resource. “They came from all parts of Europe to join with native born Spaniards, whether Christian, Jew, or Arab, to engage in the grand enterprise” (Grant 16) of translating the Arabic texts. What occurred after trying to remove Islamic culture from Europe was instead an original appreciation of Arabic texts. The collaborative effort that ensued created a community of translators, each bringing an element of his own culture. This milieu created, in essence, a cultural microcosm that by its nature accelerated communication beyond any one culture. It was an atmosphere of dialogue, of communication, and of cultural transmission.

Sometimes many languages had to be bridged to get from Arabic to Latin since the Arabic and Latin worlds had so little in common, and thus few were educated in both languages. Occasionally there could be a direct translation from Arabic to Latin. More often, though, there would be sequential translations resulting in a collaborative translating effort. "A Latin translation of an original Greek treatise may occasionally have been converted through a sequence of languages, say from Greek to Syriac to Arabic
to Spanish to Latin; or, perhaps, from Arabic to Hebrew to Latin” (Grant 16).

A great many problems arose from the translations. For instance, “significant distortion in the Latin end product from successive translations could hardly have been avoided” (Grant 16). Also, translators were not necessarily knowledgeable about the importance of the works that they were translating. This meant shorter works, that did not express the Arabic knowledge of the Greek texts, were used instead of the longer, more complicated texts that may have been better Arabic translations of the original Greek text.

Armed now with the conceptual tools needed for working with the ancient sources, and translations in hand, scholars believed that they now were able to, in the words of Bernard of Chartres, "stand upon the shoulders of the learned giants of antiquity” (Grant, 15). Following this massive translation effort in the twelfth century was a shift away from the monastic and cathedral centers of learning to the new universities that offered education to a greater number of people without the strict association with the Church. Now, education embraced the physical sciences, logic and elements of cosmology, astronomy, and mathematics (Grant 21).
Popular Mathematics during the Middle Ages

Roman numerals developed in the Mediterranean basin and were used there for centuries before they were formalized and used in the Roman Empire\textsuperscript{13}. This system was based on ones and fives (i. and v. respectively). Each power of ten had a different letter to represent the numeral; each unit power of ten had a corresponding multiple by five which also has its own letter symbol.

One reason for this numeral system's use was its pragmatic adaptation to a societal need of representational simplicity. The Roman Empire did not use paper since this had not yet been introduced to the West.\textsuperscript{14:} Papyrus from Egypt was available but because of its exotic origin it was extremely expensive. Parchment was available, yet was expensive because of the curing process in for treating the hides of either cows or sheep. and was thus cost-prohibitive. Because the major medium of representation was in stone, it was useful to have characters, both alphabetical and mathematical, of extreme simplicity and consisting of straight lines such as: I,V,X,L, and M.

The use of this numeral system was not limited to its being cut in stone. There was also an advanced system of calculation that one could employ to use the Roman numeral system in everyday life. These calculations were done with finger counting\textsuperscript{15} and the abacus. Here we become acquainted with calculation and computational methods used primarily by the populace.

Finger-counting is a basic calculation method using the fingers and the

\textsuperscript{13}I direct the reader's attention to Appendix D which contains a glossary of terms that may be useful.
\textsuperscript{14}I am not aware of when nor to what degree specifically paper was used in Latin Europe.
\textsuperscript{15}The following are apparently equivalent notation for the same calculation method and have been used almost interchangeably: finger counting or ringer reckoning, but the latter is usually more associated with determining the date of Easter.
body. Different positions of the fingers and limbs of the body would represent specific amounts\textsuperscript{16}. Not only was there a way to add, subtract, multiply, and divide but the body itself acted as the memory to the calculation\textsuperscript{17}.

Figure 2: Aventinus on Finger Symbols

"From the Abacus of Johannes Aventinus, Nuremberg, 1522 (Regensburg edition of 1532)" (Smith, History II, 201).

For example, certain patterns would be enacted to multiply different numerals; the resulting body position and the placement of the fingers and

\textsuperscript{16}refer the reader to Appendix E which contains various images of finger counting.

\textsuperscript{17}It is interesting to see that the body would be used as a component to aid in daily calculations and transactions. There is some insight given here to the society's perception of the body when the body itself would be used as a tool of calculation.
the palm of the hand would then give the answer to the calculation. Such a system was also helpful in bartering. Because of the visual nature of this method, two transactors haggling over a price could move their bodies in different gestures, placing hands at different levels and arms in certain ways to arrive at a similar price.

James T. Rogers in The Pantheon Story of Mathematics for Young People describes an example of multiplication using finger counting. (See Figure 4)

![Figure 3: Finger Counting Example](image)

"Here is how he would do a 6 x 9 on his fingers. On one hand he would hold up one finger (the difference between the five fingers on that hand and the first of the numbers to be multiplied, which was six). On the other hand he would hold up four fingers, the difference between five and nine. Then he would add the standing fingers (1 + 4 = 5) to get the tens of the answer. In this case there would be five tens, so he would know that the answer would be 50-something. To find the "something" he would multiply his closed fingers (4 x 1 = 4), and thus he would reach the answer of 54"(Rogers 70).\(^{18}\)

This calculation method was explored by the Venerable Bede, (673-735), who used this method to find the movable dates of the Christian calendar.

\(^{18}\)A similar system could be used to multiply numbers between 10 and 15. Take the problem 14 x 12. Put up four fingers on one hand and two on the other. Add them to get six; then add 10 times that sum to 100, which would give you 160; finally add the product of the fingers, 4 x 2, too reach the answer: 168. If you know algebra the equation: xy = 10 [(x-10) + (y-10)] + 100 + (x-10)(y-10) will show you why this system works"(Rogers 70-71).
year. The date of Easter, for example, could be found by implementing this reckoning system. It is likely that Bede relied on a system that was already used for centuries by the populace (See Figure 5).
Figure 4: Finger Symbolism in the 13th Century

"From the Codex Alcobatiensis in the Biblioteca Nacional at Madrid, dating from c. 1200. From a photograph by Professor J.M. Burnam" (Smith, History II, 198).
Several issues are raised by Bede’s use of finger counting. His use of finger counting shows that solving practical problems and serving religious needs were both contributing factors in developing practical mathematics. “In England, from the eighth to the twelfth centuries it was forbidden to the bishop to ordain a priest who could not compute the date of Easter and teach the method to others (Sanford, Five Fingers, 231)”. The use of finger reckoning illustrates the cultural connection and apparent dialogue that the popular and elite cultures evidently had. The mathematics of the populace gave Bede a system by which he would be able to further an intent of the Church. Because of the religious need, Bede codified the system by writing it out. In turn, this text affected the popular usage of the system by giving educators a text to serve as an example.\textsuperscript{19}

Mathematical literacy did not necessitate a written literacy. During this discussion of common mathematical usage, it is interesting to note that the ability to use the computational systems did not require the use of written language.

The finger reckoning system is a calculation method that can be learned unaided by texts. It would not take a great deal of time to learn the method from one who was acquainted with the system. In a period when memorization was a fundamental tool in education, this system could thrive. There was also no need for paper. This primarily oral system was available to anyone since there were no special tools needed for its use. Since the cost of papyrus and other media for the unwritten character was cost-prohibitive for the majority of people using it, an unwritten system was of paramount

\textsuperscript{19}Just as the dictionary influenced language, Bede’s text codified the proper use and thus affected how it was taught, heard and thus spoken by those learning the language. This example demonstrates the interconnectedness of popular and elite culture in the development of mathematics during this period.
importance to a majority of the populace.

Finger reckoning could be used by itself as an independent calculation method. In other circumstances it could be used to supplement another calculation method. One of the systems that finger-counting helped to supplement was the abacus.

![Roman Hand Abacus](image)

Figure 5: Roman Hand Abacus

"Roman hand abacus. Plaster cast of the specimen in the Cabinet des Médailles, Paris. . . Between the two rows of grooves are . . . Roman number symbols"(Menninger, 305)

The abacus is a physical calculation device.\textsuperscript{20} (See Figure 6) According to Menninger (p 306-307) the abacus originated in West. It was used by the Greeks and later by the Romans. Evidently it came into use in the East where it is still commonly used today. The medieval context of which we are speaking has its roots in the Roman use of the abacus. Here is illustrated a cultural transmission between West and East. Simple calculations could be

\textsuperscript{20}For examples of images of the abacus, I refer the readers to Appendix F.
done on the hand with finger-counting and then placed on the abacus when
doing larger calculations. The abacus consisted of vertical columns with
counting beads on each column. It was either placed on a flat surface or held
in the hand\textsuperscript{21}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{roman_calculator.png}
\caption{Roman calculator}
\end{figure}

"A Roman calculator (from \textit{calculi}, "counters") with his hand
abacus, calculating at the "dictation" of his master. From a 1st
century A.D. Roman gravestone in the Museo Capitolino,
Rome"\textsuperscript{(Menninger, 306)}.

The valuation of the beads on the columns would increase for every column
to the left\textsuperscript{22}. Often there was a horizontal bar that went across the vertical
columns. When all of the beads were pushed away from the bar, this would
be the null position. When pushed toward the bar, this would signify its
active value. Beads above this column would hold a higher value than those

\textsuperscript{21}{The abacus took different forms in both antiquity and in the Latin Medieval world. It ranged
from hand-held models to larger tables.}

\textsuperscript{22}{For a detailed explanation of the abacus and the calculations done with it, see Appendix G.}
below. The abaci with which I am familiar have beads above the bar which were valued five times greater than those below the bar on the same column. There were five beads below and when all of those were pushed up to the bar this would be equivalent to one of the beads above being pushed down to the bar and the five below being pushed down to the null position. When the column was "filled" and all of the beads were pushed to the bar, one of the beads in the next column to the left could be pushed up and the filled column could then return to the null position.

The following example goes through the steps involved in multiplying 54×26=1404.\(^\text{23}\) For further explanation see Appendix F.

![Abacus Multiplication Example 54×26](image)

**Figure 7: Abacus Multiplication Example 54×26**

It was advantageous for the abacus to be organized in this manner. With the columns associated by values of ten and with values of five related

\(^{23}\)The reader should note the first factor in this example was the product of a finger counting demonstration shown earlier in this section on popular medieval mathematics (See Figure 4).
to each unit, the Roman numeral system is emulated perfectly and thus the
abacus was useful in calculating in the Roman numeral system.\textsuperscript{24} There were
five beads below the bar (I's), one above (V's), and every column to the left
increased in value by a factor of ten. Since the board was set up in this
manner, it represented a natural progression from a numeral system based on
ones and fives and the finger reckoning system. It is important to note that
the use of a computational methods such as the abacus was dependent on the
social context from whence it came. It is likely that other cultures constructed
the abacus in a similar way that corresponded to other assignments of value
to the beads. For example, a monetary system like that in Russia,\textsuperscript{25} that had
various sets of equality for the breakdown of currency, could set up the abacus
in a way that corresponded to the various denominations\textsuperscript{26} (See Figure 9).

\textsuperscript{24}I cannot be sure when I say that the cultural influence here was strictly from the numeral
system to the calculation device. Perhaps the calculation device was constructed for the sole
purpose of emulating the Roman numeral system or finger-counting. It is reasonable to assume
that the structure of the numeral system existed before the characters used to represent it and
that the calculation device was adapted to be useful even if it may have existed prior to the
Roman numeral system in another context or culture. The issue is whether or not the abacus
existed before its use by the Greeks and Romans.

\textsuperscript{25}It should be noted that this abacus is not directly related to the western European abacus and
counting tables. Its origin is likely from China or southern Asia which had originally received
the form from the west (Smith History II 176). Once again, there is a cultural communication that
transmitted a mathematical concept!

\textsuperscript{26}With this recognition, it may increase the probability of the abacus having been adapted to fit
a specific case of numeral systems (by the Greeks). Then again, perhaps a system based on
ones, fives and tens emulating basic counting on the hand is simply a fundamental or intuitive
structure to conceptualize numbers and should not be considered to have originated with any
specific culture or period.
This example shows how the abacus was tailored to a system already used in a society. This style of abacus was likely introduced via China or a southern Russian route (Smith, History II, 176).

Karl Fink states that "the arithmetic of the abacist had for its main purpose the determination of the date of Easter" (Beman, 41). I strongly disagree with this comment. I have seen that finger reckoning was the main component to figuring the date of Easter while the arithmetic of the abacist was used mostly by merchants in the form of reckoning tables.\(^{27}\)

\(^{27}\)I have seen that when mentioned, the counting board is equivalent to the hand-held abacus, where as the counting table and reckoning table imply the larger abacus calculator devices that literally were tables.
Figure 9: The Clerk and his Reckoning Table

"A Clerk with his reckoning table and his book" (Menninger, 363).

The computational method of the abacus was expressed in various forms. Already discussed has been the abacus itself; alternatively, there is the counting table (reckoning table). The counting table appears to have been used mainly by merchants or members of the court, such as treasures, whose positions required frequent calculations.

---

28 To note the different forms that the abacus system took over the centuries, I direct the reader to Appendix H.
29 Examples of images of reckoning tables are given in Appendix I.
30 Appendix J gives the reader an explanation of the computational method utilizing the
Gerbertus, who had contributed much to introducing translated Arabic texts into Western Education, had made some new alterations to the abacus that had been used for centuries. He placed *apices* on the abacus, evidently making calculation easier, though I am unaware of the specific social causes that supported this introduction.

![Figure 10: Apices Table](image)

"A medieval multiplication table (beginning) with *apices* and Roman numerals, drawn up by a Pater Othlo at the St. Emeran Monastery at Ratisbon in the 11th century.
Bayrische Staatsbibliothek, Munich"(Menninger, 326).

Apices were symbols for numerical values that helped in some calculations, especially those that otherwise would have needed much simplification. (See Figure 11). The apices were the beginning of representing numerical values by counting table (reckoning table). The correlation to the calculation on the abacus should be noticed immediately. Examples of the abacus system are actually tables with deep grooves to hold the counter-beads.
an abstract symbol and no longer with counters that counted every element of the calculation.
Gerbertus' reckoning board, with numbered *calculi*, called *apices*. The number 705,420 is shown here. (Menninger 324)

Gerbertus did not understand the Hindu-Arabic numeral system on which these apices were based and his use of these new numerals does not express any significant difference between the new numeral system and the Roman numeral system. (See Figure 12). Nonetheless, a benefit to practical mathematics was gained by extracting an element out of an abstract system of numeration. This transition to a representationally abstract numeric system would take place to a greater extent with the Hindu-Arabic numerals.
Hindu-Arabic Numerals

In roughly the fifth century India, there arose a new system of numeration. This was a system that was based on a more abstract representation of numbers and values.\textsuperscript{31} For example, no longer would "three" be represented by something like "III", but rather by a single symbol. Each character represented a specific number. With this system also came the notion of place value. For the modern non-mathematician, it is difficult to overestimate the revolutionary nature of this notion. In systems used previously, such as the Roman numeral system, it made little difference how the letters/symbols were placed; their meaning essentially remained constant.\textsuperscript{32} With this new system, ten basic characters could be used to represent any number imagined. The only way to differentiate one number from another was the relative placement of those basic characters. The symbol for "three" if alone would represent three. But if placed to the left of a "zero," it would become the representation for thirty. The use of a symbol for "zero," the null factor, is attributed to this "new" system of numerals.

McLeish believes that: "The invention of the zero sign in particular . . . transformed calculation from a concrete to an abstract art. It made place-value as crucial an indicator of the meaning of a numeral as its physical appearance"(140). I agree with understanding that this new system introduced an abstraction. Yet, I believe that the concept behind the zero, a signification of the importance of place-value, was implicitly understood.\textsuperscript{33} The different

\textsuperscript{31}For an interesting, though biased, note on the new Indian numerals in relation to Roman numerals see Appendix K.

\textsuperscript{32}In the case of XI and IX, position does make a difference. It is the difference between 6 and 9. Unlike the Hindu-Arabic system, it was a nonessential principle.

\textsuperscript{33}Some authors have pointed to the lack of a symbol for the nullity in Western numerals. However, Western numerals were not made for written calculation; calculation was done on the hand or the abacus. It is obvious that there is a null position. In finger-counting, it is the palm upright and open and the fingers stretched forward. The abacus shows the null position with
bars on the abacus represented varying values and with the *apices* introduced by Gerbertus, the notion of number-symbols placed in various places implied different values. I feel it is the *abstraction* of the numerals that were now to represent what had been previously a more concrete and apparent expression of value that was the most important conceptual difference between the Roman numeral system and the Hindu-Arabic numeral system.

The notion of place-value was fundamental in the new Indian system. Nevertheless, the abacus can already be seen to have a sense of place value. It means something different entirely to push up three beads in the column in the far right than to push up three in the column one over to the left. Again, it would have been the difference between three and thirty. This concept was perfectly understood by those who were familiar with the abacus. It was only in the *written* form that the place-value gains its most significant difference from what was being used already in Latin Europe. It is this abstract notion of place-value that may be the issue for discourse on the revolutionary nature of the Hindu-Arabic numeral system.\(^{34}\)

The numeral system that began in India came to the Arabic peninsula in roughly the seventh century. The formulation of arithmetic, the system of calculation using Hindu-Arabic numerals, began in the ninth century in the Islamic Empire (Al-Daffá, 31).\(^{35}\) The best known text, and also the earliest which explained the Hindu-Arabic numerals, was written by the scholar Al-Khwarizmi (Levey ix).\(^{36}\) His text on this numeral system was influential

---

\(^{34}\)For some visual examples of some studies done on the physical representational transition that occurred in the Hindu-Arabic numerals, please see Appendix L.

\(^{35}\)For a short discussion on the Islamic contributors to the formulation of arithmetic see Al-Daffá pp.31-33.

\(^{36}\)A lexicon for the Arabic scholar, al-Khwarizmi, is included in Appendix M.
in bringing these numerals to Europe. Three hundred years after al-Khwarizmi wrote his text on the Hindu-Arabic numerals, it was "translated into Latin under the title *Liber Algorismi de numero Indorum*, which means the Book of al-Khowarizmi on Hindu Numbers" (Rogers, 67).

Al-Khwarizmi’s text introduced the numerals themselves, explained the nature of place-value, showed the algorithms (a word etymologically derived from “Al-Khwarizmi”) for the basic calculation functions and gave examples. Al-Khwarizmi’s text moved throughout the Islamic Empire during the ninth-eleventh centuries which at this time surrounded the entire southern Mediterranean basin. The text made its way into Islamic Spain (roughly the southern half of the Iberian peninsula) and remained there through to the eleventh century.
Figure 12: Example of Early European Numerals

"Oldest example of our numerals known in any European manuscript. This manuscript was written in Spain in 976" (Smith, History II, 75)

It appears that at least this new system was being studied by scholars since the text was copied many times over the centuries and moved such a great distance across the empire.\footnote{I am not aware of to what degree this new system had been integrated into the common Islamic culture. Since this subject does not fall into the topic of my thesis the reader should consult J. L. Berggren, Episodes in the Mathematics of Medieval Islam for discussion of this issue.}
The dissemination of Hindu-Arabic numerals texts in Christian Europe

Muslim centers of knowledge\textsuperscript{38} were well-guarded against the infidel, and offered little help to the developing West. In battles to remove Muslims from Europe, the boundary of Islamic influence was pushed further south, or from the shores of Europe entirely, as in Sicily. Great storehouses of texts that contained much of the knowledge of the Islamic empire were left behind. Latin Europe had reached a point in its development when it was avidly seeking out such texts that were to be integrated into education and that were now open to the West.

![Image: Salem manuscript]

Figure 13: The Salem manuscript

"The Salem manuscript of the 12th century is one of the oldest in the West in which computations are described with Indian numerals. 15 pages. University Library, Heidelberg" (Menninger, 411).

Beginning somewhere close to the beginning of the eleventh century, Christian Europe was in search of these texts of the ancient writers. In a search that took Europeans into Islamic libraries (whether through influence or

\textsuperscript{38} The libraries of the Islamic world were without equal and contained a veritable wealth of knowledge.
through conquest of the city that contained the library) many texts were found and copied.  

With Gerbertus' influence, texts from Spain began to be brought north in translation and elements of this knowledge were used to supplement the preexisting educational and mathematical systems of Christian Europe.

---

39 To examine a transcription and translation of an Arabic text written in Hebrew with corresponding description and explanation, I direct the reader to Levey's *The Algebra of Abu Kamil: Kitab fi al-jabr wa'l-muqabala* in a Commentary by Mordecai Finzi.
Figure 14: Al-Khwarizmi’s Text in a Latin Translation

This is one example of a page from al-Khwarizmi’s text in a Latin translation.

There were various translators and there are several surviving translations of the original text on the Hindu-Arabic numeral system.\textsuperscript{40} The European scholar, Adelard of Bath (1090-1150) was one translator who was helpful in disseminating algorisms.\textsuperscript{42} A prolific translator from England, Adelard moved through the Islamic centers of knowledge in southern and central Spain and in Sicily. Adelard was active during a great period in the development of Christian European academics, it was the opening of great centers of Islamic knowledge to the West. He had a specific interest in al-Khwarizmi’s works and translated many astronomical texts and likely the work that contained the Hindu-Arabic numerals\textsuperscript{43}.

\textsuperscript{40}There is no longer an exact copy of the original arabic text of al-Kwarizmi.
\textsuperscript{41}For a comprehensive discussion on the translating efforts of the Arabic body of knowledge into Latin and of the translators, I direct the reader’s attention to David C. Lindberg’s “The Transmission of Greek and Arabic Learning to the West”(52-90) in Science in the Middle Ages ed. by Lindberg. Pages 58-75 will be especially helpful.
\textsuperscript{42}A Latin translation of al-Khwarizmi’s arithmetic text on the Hindu-Arabic numerals was known as an algorism, which was a latinized form of the original authors name.
\textsuperscript{43}To view transcriptions and translations of al-Khwarizmi’s astronomical texts, I direct the reader to John Farvel and Jeremy Gray’s The History of Mathematics: A Reader.
Leonardo of Pisa, known as Fibonacci, also created a text that explained al-Khwarizmi’s work. When he was young he “acquired a strong taste for mathematics, and, in later years, during his extensive travels in Egypt, Syria, Greece, and Sicily, collected from the various peoples all the knowledge he could get on this subject. Of all the methods of calculation, he found the Hindu to be unquestionably the best.” (Cajori, Hist. of Math., 120).

Fibonacci became aware of the Arabic numerals early in life “and in 1202 he published a book on al-Khowarizmi’s al-jabr w’al muquabalah, in which he explains the Arab number-symbols and pointed out their great advantage over Roman numerals” (Hooper, 84). He “discussed three ways in which problems were solved in his time. One involved Arabic numerals, another the abacus, and the third finger reckoning” (Rogers, 70). It is an excellent description of the state of mathematical usage during the time while showing the mixing of the different computational methods in use.
There were various translators and several surviving translations of the original text on the Hindu-Arabic numeral system.\textsuperscript{4445}

\textsuperscript{44}There is no longer an exact copy of the original Arabic text of al-Zhwarizmi.
\textsuperscript{45}For a rather comprehensive discussion on the translating efforts of the Arabic body of knowledge into Latin and of the translators, I direct the reader's attention to David C. Lindberg's "The Transmission of Greek and Arabic Learning to the West"(52-90) in \textit{Science in the Middle Ages} ed. by Lindberg. Pages 58-75 will be especially helpful.
The computational method associated with the Hindu-Arabic Numerals

With the introduction of the Hindu-Arabic system came not only the new version of numeration (with the ten symbols and the notion of place-value) but also another calculation method. This calculation method was practiced by the algorists. Algorists were those who used the calculation method explained in the algorisms. The new method was a written form of calculation while the pre-existing forms of calculation in the West, finger-counting and the abacus, were a unwritten forms of calculation.\textsuperscript{46}

The following example works out the problem $54 \times 26 = 1404$:\textsuperscript{47}

\[
\begin{array}{c}
54 \\
26
\end{array} \rightarrow \begin{array}{c}
154 \\
26
\end{array} \rightarrow \begin{array}{c}
1364 \\
26
\end{array} \rightarrow \begin{array}{c}
1346 \\
2
\end{array}
\]

\[
\begin{array}{c}
8 \\
6
\end{array} \rightarrow \begin{array}{c}
0 \\
8
\end{array} \rightarrow \begin{array}{c}
0 \\
8
\end{array}
\]

\[
\begin{array}{c}
1364 \\
26
\end{array} \rightarrow \begin{array}{c}
1254 \\
24
\end{array} \rightarrow \begin{array}{c}
1254 \\
24
\end{array}
\]

Figure 16: Algorist Method of Multiplication: Example

Since the algorist's method was written, it necessitated a medium for its writing. Also the algorist's method was very space-consuming. A single calculation could easily take up an entire page or tablet.\textsuperscript{48} Wax tablets were

\textsuperscript{46}For a short description of the algorist's method of calculation, I refer the reader to Appendix N.

\textsuperscript{47}The problem ($54 \times 26 = 1404$) is the same as that described in the section on popular medieval mathematics when the computational method associated with the abacus is discussed.

\textsuperscript{48}For various examples of algorist divisions see Appendix O.
used, as was sand. Both were inexpensive and were easily erased; for a more permanent medium, paper was used. For the system to have been incorporated as it was in the popular culture, paper would have to have been cheaply acquired or another medium on which calculations could be done would have to be found. The European integration of this method used both the first and second. Woodblock images show paper being used and also chalk on boards.

The consideration of the influence of technology on the spread of the Hindu-Arabic numeral system is important. This specific social context would govern how readily the mathematical system could be used, certainly if paper was preferred to be used in calculations. The technology of parchment production existed in medieval Europe. It is unclear whether the growing use of the Hindu-Arabic numeral system encouraged the production of paper, or whether it was itself encouraged by greater paper production. Societal changes influence mathematics just as they might influence a growing merchant class, or would influence technology to produce cheaper paper. Therefore, it would be difficult to discern whether one element would necessarily be the cause or effect of the others. In essence, no causal relationships can be determined; only correlative relationships can be found.

---

\(^{49}\) Again I am not sure exactly when paper became available in Christian Europe nor to what extent it was used.
To study the use and impact of the Hindu-Arabic system and its corresponding calculation method on the Western European popular culture, we must not only consider some mathematical texts, but must also look at texts that would illustrate the impact this new system had on the popular culture. To begin to understand how this system was integrated and by whom, I will be looking at two Iberian texts.\textsuperscript{50} The first, a section from the \textit{Libros de moridaiides}, explains mathematics as one of the seven liberal arts and the other manuscript is a text on weights and measures that was used by merchants. The first text will show how an academic perceived Hindu-Arabic numerals, while the merchants' text will help to express what understanding there was of this system by the populace.

With a system that was chiefly oral and unwritten, it is difficult to piece together the common usage of the abacus, and to some degree finger-counting, during the period before the advent and use of the Hindu-Arabic system. To discover how the Hindu-Arabic system was used and integrated into the society, and more importantly the comprehension of the newly used system, we can use manuscript evidence. We can also note the process of transition by noting the intuitive process of the calculator, that is the person doing the calculations. By noticing how the calculator works through a problem, characteristics that express the level of comprehension of the

\textsuperscript{50}When I attended the summer study program at the Hill Monastic Manuscript Library in 1994, I participated in the \textit{Apprenticeship for Medieval and Renaissance Studies}. There, my attention was directed to a few microfilmed Spanish manuscripts. In addition to these, Dr. George Greenia supplemented the search with another manuscript on which he had been working, the \textit{Libros de Moridaiides}. It is upon these manuscripts that I will focus my attention. This explains the focus of the primary evidence on Spanish texts. Certainly these few manuscripts will not give a full nor accurate understanding of the integration of the Hindu-Arabic numeral system into the Latin West.
Hindu-Arabic system can be found. Where numerals are placed in relation to others, what numerals are written and which ones are left in the mind of the calculator can be helpful factors in describing exactly how the calculator understood the numbers and the calculation system. In the merchants’ manuscripts discussed, I will note some subtle characteristics of the author’s expression in numerals and what that tells us about how well the calculator understood the process.

*Libros de moridalides*

This text is from the fourteenth or fifteenth century, and is written in Catalan.\(^51\) The age of the text can be deciphered by one experienced in the field of paleography by examining the hand (physical method of writing characters) of the text. The portion presented, translated by George Greenia, is a description of one of the seven liberal arts; it followed a section on astronomy and came before a section on music.\(^52\)

---

Arithmetic teaches how to count and has seven manners. The first to adjust the least tally with the greatest, such as •7•

with •8• all together make •[1]5• The second shows how to take the lesser tally from the greater, which is from xxv to take x there remains xv.

The one [sic; 3rd] shows how to double, like •v• and •v• make ten. The 4th shows how to split in half, so that ten times •v• makes fifty. The 5th shows how to multiple so that as with •x• times x make

\(^{51}\)Information on this text is gathered from the notes of George Greenia’s given in the appendix and from personal relay of information from Dr. Greenia when I was studying at the HMML during the summer of 1994.

\(^{52}\)For the transcription, translation and corresponding notes for this section of the *Libros de moridalides*, see Appendix P.
C. And the 6th shows how to divide the equal portions in tallies such as when

one wishes to divide •C• by •9•1• horses that [one] taken [away] comes to •x•. And the 7th shows + {blank} such as •v• times •v• make •xxv•, from •xxxv• is •v•. And in this seventh way there are 2 ways and the first is by squared tallies and the second is by + {blank}.
The squared tally •A {blank} •A times like •v• times •v• makes •xxv•.
and •2• times three makes Nine + {blank} {blank} is the tally. •v• dos times xlv make •c xlv• and this art is counted with x• figures which are these that follow: •1••2••3••4••5••6••7••8••9••0 and the first of these letters means •one and the second two so on the following ones until •x•. ♦

It is unclear whether the author was copying a manuscript or was attempting to work out the method first-hand. In either case, the text shows that the author is totally confused by the method he describes. He haphazardly alternates unsurely between Roman and Hindu-Arabic numerals. He is unclear about the most rudimentary arithmetic. He also uses various names for the same operations throughout. When he attempts a squared tally (square), he refers to it as two times three makes nine, whereas by earlier use of those same words, the product should be six. He proves to the reader that he is unclear about the subject. For example, in the last four lines, he lists the Hindu-Arabic character "0" as equivalent to the Roman numeral "x." The mathematics described in this manuscript was considered one of the seven liberal arts. The way the author off-handedly and incorrectly uses Hindu-Arabic numerals in essentially a discussion based on Roman numerals it was
described places this new system of numeration as an "oddity." Evidently, the Hindu-Arabic numerals were only included for academic interest or as a curiosity. This manuscript shows that still nearing the fifteenth century, some learned of the Iberian peninsula were yet unfamiliar with the Hindu-Arabic numerals and their proper usage.
Weight and Measures Text

The next manuscript discussed is another Spanish text. Instead of an academic work on the liberal arts, this is a merchant text explaining how to mix wools.⁵³ The catalogue dates this text to the sixteenth century; the author is unknown. Folio 118r is found among many other similar mathematical expositions. This specifically is a description of how to properly mix four different wools, each of varying weights and prices. The mathematics involved is used to determine the best quality of mixed wool for the cheapest price.

There are -4- differences of wool, that is to say one kind that is worth [?] at -12- [§] and another [at] -21- and another 23- and another at -27- one may wish to mix [combine?] them all and that would be 200- @ what would be each worth -19- [§] ?? ?? what amount might be taken in each case.

[calculations]

One should take from them what's worth
12 [§] .83 -@ 12 36
from those worth 21 - 38 . 32 36
from those worth 24 - others- 38 32 36
from those worth 27 - others- 38 32 36

200 108 36

⁵³This manuscript is from the Hill Monastic Manuscript Library Collection (HMML #19,556). It is a microfilm of Codex Vindobonensis Palatinus MS#10585 from Wien, Österreichische Nationalbibliothek; (Vienna's National Library). This translator of folio 118R of Wien MS #10585 was done by George Greenia. A full translation of folio 118R is included along with the translation in Appendix Q.
Some Examples of calculation found on 118R are as follows:

Figure 17: Addition:

Figure 18: Subtraction

Figure 19: Multiplication

Figure 20: Division
We can note at least some degree of integration of the Hindu-Arabic numeral system into the popular or merchant classes since it is seen used in practical calculations. No longer held in the academic sphere, the Hindu-Arabic numeral system has emerged into the common usage. This emergence is at least partially due to its fulfillment of a practical need of the populace. This text shows that some merchants in sixteenth century Spain were using Hindu-Arabic numerals for calculations for a routine purpose that would suggest frequent use of the numerals.

I believe this text was to be used by others as an example. Paper probably was not yet cheap, and it is unclear why a manuscript containing only a merchant's mathematical busy-work should have been preserved unless it was to be emulated and read as a text by other merchants. It is also unclear why in earlier folios long multiplication and division problems would be worked out if the text were only to be memorized as example. I would tend to see such a work as an example which others who would use this system could study with in order to understand both the theory, method and example set by the author. This manuscript shows at least a partial integration of the Hindu-Arabic system into society. Perhaps it only extended as far as the merchant class. Perhaps it was only they who found such a system helpful. Without a comprehensive study of numerous MSS, the extent of the integration of the Hindu-Arabic numerals can only be summarized by the study of select documents that, as representative, should give an idea or impression of the transition at different periods in various sectors of the population.
Similarities Between the Abacus and the Algorist’s Method

During my study of this topic, I took the time to learn how to calculate using the abacus and the algorist’s method. Though I was not able to become proficient in either of their usages, I was able to find some comprehensive similarities between the calculation methods discussed above. It is important to remember that the Hindu-Arabic system fell onto a continent that had a pre-existing numeral system and its own calculation methods. What we find therefore is a dialogue between the old system and the new.

There are some striking similarities between the calculation method using the abacus and the calculation that uses the algorist’s method. The treatment of the numbers had the same "feel." Numbers with which we are today familiar “feel” independent from one another. When added together, they stand whole until the final product is reached. When one adds or works in most any operation with the abacus, the numbers once placed into the system are "lost." When two numbers are added on the abacus, the numbers "become" the sum. With the abacus, it is nearly impossible to go back and check to see if the work is done correctly. It is also very easy to lose track of the calculation; since there is nothing to remind one of the step being done in the calculation, one must often start over if distracted from the work.

If the reader had the opportunity to work through the examples in previous sections or has an opportunity to work through the examples in the appendix of how to calculate using the abacus and algorist methods, the pattern of similarity that I will describe would be more evident. I will note the similarities between the two methods in two operations, addition and multiplication.

Addition with the abacus immediately processes the addends to form
the sum.

Figure 21: Abacus Addition

In the manuscripts with algorist calculations that I have studied, I saw that the addends no longer remained distinct when placed in the calculation. We see an associativity of digits placed in the same place-value column. A string of addends such as \(63 + 4 + 31 + 8 + 24\) would appear like this in the algorist's calculations:

\[
\begin{align*}
63 \\
34 \\
21 \\
8 \\
4
\end{align*}
\]

The addends seem "fluid." They are not independent and flow together in the

\[54\text{Although the notion of place-value is asserted to be revolutionary by many secondary sources, I see that this process appears to look much like the addition done on the abacus.}\]
process of finding the sum. Perhaps this expresses an implicit comprehension of the issue of place-value; it could also be expressive of a notion understanding the associativity of terms in similar value places, as is used when calculating with the abacus. The similarity of implied associativity can be seen then between the abacist and algorist methods.\textsuperscript{55}

Multiplication in both of these methods also holds a great deal of similarity. Each element of the multiplicand is taken independently and is acted on by the multiplier. The individual components of the product, in the case where the multiplier is more than one character, are added to the existing answer to form a comprehensive sum. The sum is accumulated until all of the elements of the multiplicand have been multiplied out. They are both similar processes where the partial products of the multiplication are continually added to the working sum.

The following figure pulls together the multiplication examples worked out in previous sections (See Figures 8 and 17) so as to note the similarities found between each.\textsuperscript{56}

\textsuperscript{55}I noted this associativity of similar place-value terms while studying the marginal calculations done in the Wien MS #10585.

\textsuperscript{56}When I was studying the two systems of calculation, I was instantly impressed by their similarities.
Figure 22: Abacist/Algorist Similarity

"Now let us look back again. The old calculations on the abacus and the
"new" operations with digits, crossing them out or erasing them as they are finished with - we are surprised to realize that they are essentially the same. The operations with digits merely translate the steps on the counting board into written calculations on paper or sand tablet" (Menninger 331).

That there was such a great similarity between the calculation methods was fundamental for a transition from one established cultural system to another cultural system's influence. I feel that this similarity helps express the process of transition that occurs when a new system finds itself introduced into another culture with a pre-existing system. There is not a complete break from the traditional method. Elements of the new method find similarity to the pre-existing method since a system is more easily adapted when a culture finds the newer method familiar.

I believe that there are three reasonable explanations that may account for the similarity of the systems. The first possibility is that the similarity could arise from the influence of the old system on the new, changing the newer system and melding it to become more like the old and therefore more easily used by those acquainted with the older system. A second explanation is that of all of the possible calculation systems that may have taken hold (although I am not aware of any others), this is the one that people found comfortable enough to use since it was so similar to the system in use.

A final hypothesis considered could be an even easier explanation of the similarities between the systems. Perhaps the human mind calculates in a certain way that it finds natural. According to this theory, when the two systems, European and Eastern Indian, originated independently from the other they followed patterns that were natural for comprehending numerals and found calculation methods that incorporated this common nature. Although yet on a continuum of change, the different calculation methods
associated with different numeral systems appear to have branched out from an original commonality. I believe the similarity of the systems can be described by a combination of the three possibilities.
Figure 23: Arithmetic’s Blessing of the Digits

"The old and the new are symbolically represented in the book by Gregor Reisch: Next to Pythagoras with his sorrowful face working at his reckoning table sits the cheerful and serene Boethius contemplating his computations in the new numerals. Arithmetic, personified, hovers with her books between them, looking at the computer with the digits and indicates her approval of him by the two geometric series in Indian numerals on her garment" (Menninger, 431).
As the algorist system spread across Western Europe, it came into conflict with those who were using the pre-existing abacist method of calculation.\textsuperscript{57} The introduction of Al-Khwarizmi's text and its use of Arabic numbers and the zero "became the centre of a three-century-long ideological battle in Europe for and against the new arithmetic" (McLeish, 140).\textsuperscript{58}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure24.png}
\caption{Computing Controversy}
\end{figure}

"The title page of Adam Riese’s *Rechenbuch* (1529) illustrates the computing controversy of his time" (Swetz, Capitalism, 32).

\textsuperscript{57} For various images illustrating this controversy, I refer the reader to Appendix R.

\textsuperscript{58} I do recall reading, though am unable to remember the citation, that texts written with Roman numerals were taken as stronger evidence than were texts written in Hindu-Arabic numerals.
This controversy is seen in various woodblock prints that are found at the
beginning of the algoritms that explained the Hindu-Arabic numerals. There
was a difference of opinion as to which system should be used. Some people
felt that the new algorist system was far simpler and the calculations were
easier. Others felt that the Roman numerals and the abacist calculation
method were perfectly adequate.

A difficult element in studying the controversy that centered around
the varying levels of acceptance and use of the two calculation methods is the
disagreement found in secondary sources. Sources claim various societal
groups, for example merchants and academics, responded to the introduction
of Hindu-Arabic numerals into Western Europe in different and even
opposite ways. Some sources claim that academics encouraged the integration
of the Arabic numerals while other sources feel that the academics did not
have faith in these foreign characters and the algorist calculation method.
Similarly, secondary sources disagree on how the merchants accepted this
numeral system.

I find that it is not possible to simply label one sector of the European

---

59 For a discussion on the different sectors of society and how they were thought to have
responded to the introduction of the algorist system, please refer to the next section of the thesis.
60 Another fact that makes this study difficult is again the obvious bias of the secondary sources.
This is apparent through such words as: deprived, struggle and victory. Fink states that: “The
abacus, the heir to the computus, i.e., the old Roman method of calculation and number-writing,
was destined to give way to the algorism with its sensible use of zero and its simpler process of
reckoning, but not without a further struggle” (Beman, 39). Menninger tells us that the Hindu-
Arabic numerals “do away with the abacus and calculi, but only by introducing the zero into
the writing of numbers. One might say, in a nutshell, that the zero overcame the abacus. But
its victory, which started in the early Middle Ages, took a long time” (331). “In actual fact
Europe adopted [Arabic numerals] from the Muslims only in the thirteenth century. Fighting
their introduction and that of the decimal system that went with them for several hundred
years, Europe deprived itself of the advantages of one of the world’s greatest contributions to
mathematics” (Al-Daffa, 33).

61 McLleish, in Number, claims that “opposing change was the majority of merchants and
accountants, accustomed to the use of the abacus and of Greek and Roman alphabetical
numerals” (140). Yet Swetz, in Capitalism and Arithmetic, points out that it was the merchants
who were the fundamental force behind the transition (291-295).
medieval and Renaissance society as "for" the incorporation of the Hindu-
Arabic system and its corresponding calculation method and another sector
"against." Relegating the influenced and influential sectors of society to either
side of an artificially imposed bi-polar issue would simply be an over-
simplification of a multi-influenced, transitory cultural situation. To look at
the different trends that occurred within the society, and to understand who
influenced the introduction of the algorist system into Europe, I take this
discussion to the next section.
Reasons for the transition from Roman Numerals to Hindu-Arabic Numerals

Swetz, in *Capitalism and Arithmetic*, claims that the societal events of the fifteenth century were influential on the transition from the Roman numeral system to the Hindu-Arabic numeral system. Social transition in the economic and intellectual world affected the course of mathematical use. Swetz claims that the growth of mercantilism was influential in the transition to Hindu-Arabic numerals.\(^{62}\)

During the fifteenth century Italian Renaissance, as cities were expanding with commercial activity, people were questioning sacrosanct ideas and concepts that led to new inquiry and innovation (Swetz 4). It is during this same period that the abacus with its counters began to disappear from Spain and Italy (Cajori, History, 122).\(^{63}\) With evidence gathered from the *Treviso Arithmetic*\(^ {64}\) and with a recognition of the social transition of the early Renaissance that was occurring simultaneously, Swetz points out that: "Europe's mercantile development from the thirteenth century onward placed increased importance on an understanding, and proficiency in, commercial arithmetic"(Swetz, 15).

---

\(^{62}\)Swetz's work identifies the general trends that affected both society and mathematics. His description of the influences of different areas of the society to one another presents a much different historiography than that of Cajori, McLeish and Menninger who each declare a sector of society that did and another that did not accept the new system.

\(^{63}\) "In France it was used later, and it did not disappear in England and Germany before the middle of the seventeenth century" (Cajori, History, 122) Cajori's citation is taken from George Peacock, "Arithmetic" in the *Encyclopaedia of Pure Mathematics*, London, 1847 p.408

\(^{64}\) This is the earliest popular printed arithmetic. This 1478 practical gives insight into the social influences on mathematics in the early Renaissance. For further information on the *Treviso Arithmetic* and for the resources that were found to be influential in the study of this arithmetic, please see Appendix S.
"During the twelfth through the fifteenth centuries, algorisms appeared in great numbers and in a diversity of languages other than Latin" (Swetz, 29). This is evidence that since texts describing Hindu-Arabic numerals were
being translated into the vernacular, the numeral system was finding common use. "From the thirteenth century onward, thousands of students at the universities had a Latin exposure to the new arithmetic, but it was not until the acceleration of commercial activities and the advent of printing that this knowledge was disseminated to the 'common man'" (Swetz 33).  

Figure 26: The Son Apprenticed to the Abacist

"A father apprenticing his son to a calculator, who appears to be fully in the camp of the abacists" (Menninger, 435). The woodcut is by Hans Weiditz.

Middle class sons could be brought to "reckoning schools" to learn the correct calculation methods. The text Swetz studied reveals that "the Treviso Arithmetic is an algorism, a practical, intended for self-study and relevant to the commercial and reckoning needs of Treviso and Venetian trade" (33).

65 To study a reproduction of an instructional arithmetic printed in English with corresponding sources and notes, I direct the reader to Struik's Source Book in Mathematics, 1200-1800, Section 2, pp.4-6.
"Mathematics was moving from the realm of scholastic speculation to the application of manufacture and the marketplace" (Swetz 34).

Mercantile activity was becoming an integral part of the common life in fifteenth century Italy. An element of this activity was money and money-changing. Because the exchange of money was becoming integral to the livelihood of the populace, some extraordinarily stringent penalties were given to those who debased currency in the 14th century Venetian Republic (Swetz 273). As trade expanded, standardization was an issue for exchanges. This accelerated the need for exchange houses and numbers of exchange rates. A practical need for exchange rates necessitated greater understanding and usage of mathematics.

Also, "In so far as history can ascertain the Hindus were entirely original in creating one other idea which proved immensely important later on. This was the concept of negative numbers...[which] could be...useful...by employing them to represent debts" (Kline, 93). Essentially, "In arithmetic, commercial expansion led to the development of new and more effective methods of computation" (Scott, 84). Again we see pragmatic reasons for society's effect on the popular use of mathematics.

Another reason given for the rate of incorporation of Hindu-Arabic numerals into the Christian West is offered by Linn. "There may have been some pressure exerted by the Church. After all, there was a choice between the numerals from the center of Christendom and the symbols invented by the infidel Moslems, or the Jews, as many people believed ... and it would not be surprising that their ideas should be dimly regarded by the persecutors" (120).

---

66In Venice, "in 1321 provisions were made to inspect money changers' benches to insure against debasing; in 1357-1359, the penalty for this offense was the cutting off of the right hand; those caught coining were physically disfigured--women had their noses cut off and men were blinded" (Swetz, Capitalism, 272)
James T. Rogers also notes some of the reasons for the rate of incorporation of the Hindu-Arabic numerals into Western Europe. He notes that the transition was likely slowed by the expense of paper. For this reason, finger counting and the abacus would remain popular. Swetz disagrees only slightly in noting that, at least in Italy, there was an "available paper supply"(Swetz, 26). Nonetheless, Rogers goes on to state that printing helped to move ideas. With the spread of the printed algorismic texts, Hindu-Arabic numerals were made available to the populace, and thus the use of the algorist method spread.

Rogers continues with an interesting note on a relationship that one culture would have on another. With the fall of Christian Constantinople to the Turks in 1453, learned Greek scholars who were well-acquainted with the Hindu-Arabic method "fled to Europe ... like seeds falling on fertile soil"(73). This movement of scholars from Eastern to Western Europe is expressive of how cultural interconnectedness would in turn foster a form of cultural communication.67

67 Swetz offers some additional interpretation as to why the the Hindu-Arabic numeral system and the algorist method would be appealing to those who decided to use it in the early Renaissance and why some of the merchants decided to reevaluate their calculation method. For this discussion, I refer the reader to Appendix T.
Conclusion

This thesis has identified the transition between the Roman numeral system and the Hindu-Arabic numeral system during the later Middle Ages and early Renaissance. This numeric transition was expressive of a greater cultural transition occurring simultaneously. The newer numeric system was shown to be related to the preexisting calculation method thus showing the interrelation of the various mathematical concepts a society employs.

I have discussed various medieval mathematical systems and their corresponding calculation methods after providing a historical context for this thesis. I have noted the popular use of the system of finger counting or finger reckoning, the Roman numeral system and its corresponding computational methods, the abacus and the reckoning table. I have used this discussion to prepare the reader to note the transition to the Hindu-Arabic numeral system. I have also noted the people and texts that influenced the algorist system’s incorporation into Europe.

Using manuscript evidence, I identified the similarity of the older and newer computational methods. I then made some hypotheses about the connections between the transitions that were occurring in the greater society and those transitions that occurred between numeral and computational systems used by the populace. Exactly how does a culture "decide" what to use? What are the reasons for transition from one socially accepted numeral system to an entirely new numeral system? I have offered a few suggestions while showing the interconnectedness of the societal concerns that were influential.

There were social influences that affected the transition from one type of numeral system to another. The changes, especially noted with the rise of
the merchant class, that were happening in social and economic aspects of society at the onset of the Renaissance pushed the common use of the mathematical expression of that same society to change. Many of the secondary sources, on which I had to rely, did not seem concerned with how fully a particular calculation method had been incorporated into or accepted throughout all levels of culture. Some of those reasons for the numeric transition that were affected by the surrounding society which itself was in transition have been discussed.\textsuperscript{68}

I have noted the connections between different cultures to the factors in question in this thesis. Once I established that this Western European numeric transition originated from societal factors, various other cultural influences were found to contribute. There was an influence, as seen with the abacus, from West to East. A reciprocal dialogue between popular and elite was seen especially with finger counting. Finger reckoning also expressed the relation of the secular use to the clerical concern. There was also a dialogue, however hesitant, especially between the Christian and Islamic societies.

I found some difficulty in my research when navigating through biased perspectives of secondary sources. The Middle Ages, and specifically its mathematics, should be understood in contemporary and appropriate terms without modern disregard.\textsuperscript{69} It is important here, as with any culture, to identify and understand each sector that comprises a society in order to gain a

\textsuperscript{68}For a wonderfully concise and expressive note on the interrelations of mathematics to the society in general, please see Appendix U.

\textsuperscript{69}I feel that James T. Rogers in \textit{The Pantheon Story of Mathematics} expresses well the state of medieval mathematics, and society in general, when he states that the men of the Middle Ages "tended the garden of mathematics" (Rogers 65). It is an excellent picture of wealth and bounty enjoyed in all areas. It should not be forgotten, though, that this garden, too, produced fruits. This bounty is not always recognized by those searching for theoretical ambrosia. Also, a reciprocated cultural dialog is occurring, though not always recognized between the elite and popular cultures. Often, unknowingly, the cultures borrow from one another and actually rely on the other for new ideas, possibilities, and applications for those ideas.\textsuperscript{69}. 
full comprehension. It is important not to fall into a non-contemporary over-emphasis of a single element of a culture. Emphases of the pure or elite components found in modern histories are examples of this. All elements of a culture must be understood in context.

This thesis has offered insight into one element of the Medieval and Renaissance world. It is an understanding of this one elements of a culture that was re-prioritizing and re-evaluating that will be helpful to understand the cultural transition from which the mathematical transition was influenced.
Figure Bibliography

Winged Arithmetic .................................................. Menninger 431
Aventinus on Finger Symbols ..................................... Smith History II 201
Finger Counting Example .......................................... Rogers 70
Finger Symbolism in the 13th Century .......................... Smith History II 198
Roman Hand Abacus ................................................ Menninger 305
Roman calculator .................................................... Menninger 306
Abacus Multiplication Example 54x26 ........................ Pekarek
Russian Abacus ...................................................... Smith History II 176
The Clerk and his Reckoning Table .............................. Menninger 363
Apices Table ........................................................ Menninger 326
Gerbertus's Apices ................................................... Menninger 324
Example of Early European Numerals .......................... Smith History II 75
The Salem manuscript .............................................. Menninger 411
Al-Khwarizmi's Text in a Latin Translation .................... Rogers
The Carmen de Algorismo .......................................... Menninger 412
Algorist Method of Multiplication ............................... Pekarek
Wien #10585 Addition .............................................. Trnscp. by Pekarek
Wien #10585 Subtraction ......................................... Trnscp. by Pekarek
Wien #10585 Multiplication ....................................... Trnscp. by Pekarek
Wien #10585 Division ................................................ Trnscp. by Pekarek
Abacus Addition ..................................................... Pekarek
Abacus/Algorist Similarity ....................................... Pekarek
Arithmetic's Blessing of the Digits .............................. Swetz, Fingers 261
Computing Controversy ............................................. Swetz, Capitalism 32
The Merchant and his Counting Board ......................... Menninger 363
The Son Apprenticed to the Abacist ............................. Menninger 435
Geometry of Gothic Church Windows ........................... Swetz, Fingers 228
Boethius Finger-Counting ........................................ Dedron 166
Bede's Finger Mathematics ....................................... Swetz, Fingers 220
Finger Reckoning ................................................... Taton xv
The Art of Finger Reckoning ..................................... Linn 105
Finger Symbolism in the 13th Century .......................... Smith, History II 198
Pacioli on Finger Symbolism ..................................... Smith, History II 197
Annotated
Bibliography
Sources


Al-Daffá states that his work "presents the Muslim contribution to mathematics during the golden age of Muslim learning approximately from the seventh through to the thirteenth century" (7). It is an excellent overview which appears well-aware of its historical context. This text discusses how mathematics was formalized and simplified by Muslims during the European Dark Ages and later introduced to Western culture (in Preface, 7). Yet, terminology such as "European Dark Ages" suggests the historical bias implicit in the work.


This work has an obvious bias against cultural relativism and the appreciation of the popular medieval mathematics discussed in my thesis. There is a discussion I find rather inappropriate that works to illuminate my hesitancy to use this source. Bell states as follows: "Beyond the cumbersome Roman numerals, which can be called a mathematical creation only by undiscriminating charity, the Romans created nothing even faintly resembling mathematics. They took what they needed for war, surveying, and brute-force engineering from the Greeks they had crushed by weight of arms, and were content." (79) A final statement that would be the epitome of my concern is the comment that "...European civilization rotted..." (80)

This work discusses arithmetic from the eighth to the fourteenth century with some additional relevant comments found in the section discussing the fifteenth to nineteenth centuries. It is a rather oversimplified explanation of the means and processes by which the various calculation methods were used in the Middle Ages. Often the wording is over-generalized or is obviously biased as shown by: "... dominated ... destined ... sensible ... struggle..." (39).


This work discusses the Arabic translation efforts to incorporate Greek knowledge. This work is careful to recognize the influence of texts and mathematicians on the greater Islamic society. The incorporation and common use of various mathematical modifications are considered. Although not entirely on the subject considered in my thesis, this work is an excellent source for the reader to understand Islamic medieval mathematics and its Islamic social context.


The period Butzer discusses (AD 400-950) is a little earlier than the intent of this paper, yet the "Table of Scholars and Works in Science (AD 400-950)" pp. 599-605, and "Mathematics in West and East from the Fifth to the Tenth Centuries. An Overview." pp. 443-481, both by Paul L. Butzer, may be helpful to understand some important persons, texts and trends in the years leading up to the period discussed in my thesis. His research appears thorough and conscious of the elements that composed the trends of the period he discusses. "My overview attempts to show how these many and diverse components, directly or indirectly linked to the mathematical and astronomical science, fit into
a broader vision of the whole" (Butzer, 441).


The chapter on Europe during the Middle Ages has some helpful exposition on key scholars. Included is a discussion of some Arabic works translated into Latin, the authors of the translations and the reaction to these texts in the West. The research is dated and the style of the text appears somewhat judgmental, such as the following: "The new notation was accepted readily by the enlightened masses, but, at first, rejected by the learned circles" (121) Firstly the word enlightened itself is anachronistic to the period being discussed. Secondly I see that this is virtually the opposite conclusion to what was reached by McLeish's and Menninger's scholarship.


This text spends a great deal of time examining certain manuscript evidence to discuss the transition witnessed by the physical representation of the numerals themselves. It is a study of the notational history of Hindu-Arabic numerals whereas my thesis discusses the influence of those numbers within their societal context.


For one interested in viewing translations of other texts from the same period as discussed, though not directly associated with my topic, this is an excellent source. It presents "famous" texts, yet contains little to no information on the cultural usage of those works.

By its nature, this work follows the influence of Archimedes and the ancient works on the general science and specifically the geometric treatises of the Middle Ages. With ample documentation, manuscript evidence and illustrations, this text is extensively thorough on the proposed topic. Nevertheless, while the subject was tertiary to my thesis topic, it was still able to offer interesting information on critical readings of medieval manuscript evidence.


This source was used for its reference to Adelard of Bath. The reference notes the Arabic influence on Adelard’s translating work, the various texts attributed to his translating efforts and their influence on Western Europe. Included are primary references to texts and some criticisms of secondary sources that have examined Adelard’s contributions.


This work is a collection of commentary on the more recognized texts and the contemporary mathematicians who dealt with them. It does not discuss the integration of mathematics into the greater society, nor does it appreciate society’s influence on the mathematical texts discussed. The works are treated as if they appeared in a vacuum of culture and are discussed as important without identifying their relative importance to medieval society.

Joseph W. Dauben's chapter: "The King was in His Counting House..." contains a concise discussion of the Darius vase, the Salamis tablet and the early history of the abacus system. In another chapter by Dauben entitled "The First Digital Computer: A Handy Guide to Simple Finger Counting," hand counting is described. There is also mention here of various other systems of numeration that are not of base-ten. The chapter "Zero: The Exceptional Number," by Stephanos Gialamas and Miriam K. McCann, discusses some cultures' understanding of the nullity and their expressions describing the same. The chapters are general overviews of the topics mentioned. The writers are aware of cultural integrity and the various contributions made by cultures in the realm of popular mathematics.


This text briefly mentions the history of finger-counting. There is a discussion of notable translators, their contributions and influence on mathematics during the Middle Ages. Also included is a description of the "Jewish communities of Spain and Provence,... [that led] to a slow revival of the intellectual tradition of Antiquity"(170).


This article discusses the relationship of mathematical images and analogies to subjects outside of mathematics, especially to theological problems. It explains how mathematical concepts were used to explain
theological ones, and vice versa. The article illustrates well how closely the world of mathematics was united with the theological world.


Evans' article discusses Adelard's text: the Regule Abaci. Adelard's text is a treatise on the abacus meant to explain the use of this method of calculation. The various elements of the text are explained well. Different sources are noted as are their corresponding variations in the text. This work carefully recognizes the various influences on Adelard and notes what can be learned from Adelard's text.


Texts of Boethius, al-Khwarizmi, and Fibonacci are presented in translation. Although interesting, al-Khwarizmi's astronomical texts are not directly related to the focus of this paper.


This work discusses the state of science in medieval Western Europe. It looks to social influences on science, yet appears to be searching for a justification for an implicit claim that all intellectual life had crept to a halt during the medieval age. The popular usage of science, especially mathematics, does not play a role in his argument. Nonetheless, for a general understanding of the medieval period in science, this source offers a good start to the reader.

There is short mention here about numerology in mathematics during the Middle Ages. The text mentions numerology only in its relation to natural philosophy and gives little explanation of the former subject. "Despite the abundance of numerology in medieval theological and devotional literature, it made curiously little impact on the major works of natural philosophy"(56). At best, this work is only mildly related to the subject of this thesis.

Greenia, George. Unpublished transcription and translation of *Libros de Moralidades*.

Included are: two pages of introduction, a transcribed folio, its translation, and endnotes. Summer 1994.


Greenia’s work on folio 118r was specifically done to assist my study during the Summer 1994 program at HMML.


There are some good color images, especially of selected pages from manuscripts.

This work lightly touches on and briefly discusses different trends and the contributors during various periods. It tends to be too heavy with names and dates due to the author's conciseness. It seems too broad an approach for the reader to understand the elements that would influence the movements of trends during the Middle Ages.


The given overview of periods in the text seems almost too general for one to recognize the specific elements of transition or to justify the arguments given. The presentation of evidence appears to skim over the topics too lightly to give the reader the information necessary to understand the period discussed.


Jones notes that finger symbolism was “once used to communicate at international fairs and as an aid in remembering numbers when using the abacus” (359). The short chapter gives a brief summary of calculation methods that could be used independently or as supplements to other methods. The algebra that explains finger counting is mentioned.


This text accounts for the Western medieval period as a reversion into
ignorance promulgated by Christian faith. Man's desire for salvation and the after-life is given as the death of reason. It is Kline's conjecture that reason was supplanted by religion (92). The Church is described as battling to erase the pagan mythology from people's minds while recognizing nothing of popular religion's integration of doctrine and mythology. There is simply a negation of any popular use of mathematics during the Middle Ages.


Levey states that his motivation for his work on Abu Kamil is because: "Following [the] foundation work of al-Khwarizmi, the earliest Arabic algebra which is known to be extant is that of Abu Kamil" (ix in forward). The text gives the transcription of the Hebrew text and English translation with description of Abu Kamil's contributions in relation to al-Khwarizmi, the proofs and examples of the text and a discussion on the influence on the Islamic golden age.


This chapter carefully explains the flow of Greek knowledge from its dissemination through the Islamic Empire to its transmission to the West. This chapter also describes in detail the translation efforts.

This is an excellent work on the social context of which science was an element. The dialogue between science and various societal elements plays a key role in understanding how science was understood and how it was integrated into the society in general, in both Arabic and European context. The discussion is a basis for the reader to understand the trends that influenced both science in general and, in particular, mathematics. It portrays the medieval period as an active contributor to the advancement of science, yet in this section begins to rely on "famous" texts to substantiate the claim without looking to society to show any popular use of mathematical methods.


With a chapter entitled: "The West's Asleep," referring to AD 500-100, I see the possibility of some prejudice held against the common mathematics of the period I discuss. Alternatively, explanations of broader cultural trends are described simply and concisely. The reader may find this work helpful with understanding some general themes associated with the period.


This is an clearly written chapter describing the transitions between and similarities of the different computational methods used during the Middle Ages. The later chapter goes on to note the Euclidean geometric and algebraic developments that occurred. I started with this chapter in my research and I feel that it has given a good background while explaining the abacus, the algorist method, and influential figures of the period.

This text is a careful summary of the influence of al-Kharizmi's work on both Arab and European mathematics. Yet the following comments should be noted. "Euclid and other Greek mathematicians had freed geometry from the shackles of land surveys and building problems, enabling scholars to think about the abstract properties of space. The Arabs did a similar service for the number"(140). The Arabs "made arithmetic so simple,"(140) yet the "majority of merchants and accountants"(140) opposed change. What can be inferred from these comments is that the common users of the older method chose to remain ignorant of a simpler and better system to remain in mathematical "shackles" even though the academic elite offered emancipation. I could not find in this text any appreciation of the popular mathematics of the medieval period. I found that Cajorli's *History of Mathematics* found the opposite conclusion. Rather than the common users opposing the change, Cajorli suggests it was the academic elite who withstood change.


I feel the source is compromised by seeking out the roots of modern numerals and not understanding the context in which the numbers changed nor the impetus for the development of the numeral system sufficiently. It is a rather confusing amalgam of dates and evidence that does not hold to the very topic headings under which the information should fall. The discussion is anachronistic and wandering. The vast and comprehensive research, through numerous examples, is unparalleled yet it is masked by a biased view that considers steps toward modern Western mathematics as advancement and all else as a road-block to progress. Consider his discussion on the Hindu-Arabic Numerals: "after a thousand years of migration from
their initial home, their ultimate victory was assured. This was the victory of an alien culture, to be sure, but also a victory for the mind of man, who finally, in the long history of written numerals, arrived at a mature, abstract place-value notation: the Indian numerals, which have become our own, were now fully developed" (Menninger, 445). No longer in print.


Although this text over-simplifies the intricacies of the common usage of mathematics in the Middle Ages, it describes Arabian influence on Western mathematics well. It "offers vistas - sketchy as they must be when written for young people - of large and splendid regions which promise satisfaction and adventure" (Morris Kline's introduction, 7).


This chapter gives a brief history of the loose counter abacus. A description of calculations done by this method is reproduced in Appendix I.


Schrader’s chapter describes the evolution involved in creating the unique emphasis shown to mathematics during the Middle Ages in the Universities. The seven liberal arts are discussed, since mathematics is one of these, as is number mysticism, the influence of Arabic texts in universities and the curricula taught at those
universities.


The language used in this work portrays various elements of the medieval period as warring or battling with each other for a victory claimed by one or the other at any given point. Medieval Europe is expressed to be in the middle of "...intellectual stagnation..."(53) having never risen above mediocrity. This work does not allow for the dialogue between cultures, the mix of different methods and how the changes in those methods influenced a culture's development.


Awash with names and dates and deeds, this textbook mentions the major texts and mathematicians. The sections of the text are separated according to century, country, and contributor. This makes the book difficult to follow and buries the general trends. It should be noted that the author, as stated in the introduction to Vol. II, intended for this volume only to whet the students' interest and to "encourage him to pursue his reading"(viii) elsewhere. It is an incredibly difficult text to use; it is on par with the Sarton reference volumes.


The third chapter: "Mechanical Aids to Calculation," would appear to be most topically associated with my thesis and itself is a useful source. Yet the first two chapters: "Development of Arithmetica", and "Logistics of Natural Numbers" contain information perhaps even
more related to my thesis. In these chapters addition, multiplication and division using the algorist method are discussed. Like the first volume, this text can become confusing. Similar to Menninger, Smith describes algorist "galley" division before even discussing the abacus and finger-counting. The text is concise, well-documented, and easy to read. Although the research and writing is excellent, the organization of work makes it conceptually difficult to follow; the general trends are hidden by the topical organization. Others have specifically noted Smith's work, though, as being fundamental to preliminary research in the topic of the history of mathematics.


The first selection in the chapter on arithmetic is Leonardo of Pisa's "rabbit problem" describing the Fibonacci sequence. What follows is a selection from Robert Recorde's (c.1510-1558) work on arithmetic; it is the first arithmetic printed in English. The reader may find interest in a reproduction of an early instructional arithmetic printed in English with corresponding sources and noted.


This source is especially useful in describing the trends, especially capitalism, that affected the transition from the Roman numerals to the Hindu-Arabic numerals. This work, spurred by the discovery of the *Treviso Arithmetical*, which was translated by David Eugene Smith, discusses the societal influences of the Italian Renaissance expressed in this mathematical text. He discusses how the increase in mercantile activity spurred the learning of reckoning methods; he summarizes the various types of texts being printed; he offers a brief etymology and
short history of the abacus; he also discusses the various translations of al-Khwarizmi. The author is aware of the cultural integrity of each social group that is discussed. The work shows the societal trends apparent in the text studied without rendering judgments about the acceptance of Hindu-Arabic numerals. Although the specific case of the societal influences of the Italian Renaissance felt in the city of Treviso is discussed, the trends are meant to apply to Christian Europe in the general case.


Swetz’s introduction to the section, “European Mathematics During the ‘Dark Ages,’” is an excellent overview of medieval society and how it should be understood and respected on its own terms. The chapters Swetz assembles for this text display various aspects of popular and elite contributions during the history of mathematics, both in the East and the West. The selections by various authors are well written and easily read. Swetz’s own “Historical Exhibits,” which are structured around images related to the period, give interesting insights to the topics discussed. Together, this work combines seemingly disparate elements into a history that gives the reader a fuller comprehension of any given period studied in the text.


Although this work is carefully divided into cultural, temporal and personal contributions, it is difficult to follow. Its focus is certainly on

67 Swetz claims the origin of the abacus was in the West.
science, yet due to its precision, this work gives an excellent context for the place of mathematics in the medieval sciences to the reader.
References


The references given barely fulfill what would be required of a dictionary. It expresses a rudimentary knowledge on my related topics and persons. No sources for the references are given. The very basic writing is almost too general to help further research.


Included here are some preliminary musings on the philosophy of the history of science. The timelines presented here are helpful, yet the rudimentary outlines of mathematicians give only "notable" events or works. This work does not seem to be concerned with relating the social context in which these contributors worked and how these contributions affected society.


This work gives concise descriptions of "famous" contributors which are helpfully cross-referenced. The referencing accentuates the connections of the Western scholars to their Arabic influences. Sources for each reference are cited; this is appreciated when conducting initial research.

This is an excellent annotated bibliography giving sources from various languages. It appears quite extensive and should be helpful for the reader who is interested in a topic similar to that discussed in this thesis.


This work contains helpful references with sources cited for each reference. The entries concisely express not only the respective contributions, but also what influence was seen in Western Europe by those who contributed.


This Spanish source offers a clear and well-organized table that highlights the transitional stages of the physical appearance of the Hindu-Arabic numerals as they changed over the centuries. As I noted in the text accompanying Figure 50, much of the work on Hindu-Arabic numerals has been on their *physical* transformations through the centuries.


This gives an index to the arithmetic contents of Migne.

293-298. Turnholti (Belgium).

This appears to be Bede's text on finger-reckoning. Another text must be referenced to be sure the texts attributed to Bede by Migne are truly from Bede.


This source is extremely thorough yet relatively unusable. It is broken down into centuries and then into subject headings; each is then discussed in turn. The general trends that influenced mathematics, for example, are not readily apparent upon reading this text. The popular experience as an element of the history of a society does not appear here. For a study of a specific period and the individual contributors of that time, this work would be helpful.
Manuscripts

Wien MS #10585; HMML project #19, 556. XVI century, 163. Quarto.

Spanish weights and measures containing algoristic division. It is a
treatise on arithmetic’s use for the mercantilists. “Tractatus hispanicus
de arithmetica praesertim in usum mercatorum” (Wien catalogue, 209).

Wien MS #12600; HMML project #19, 993. XII et XIII centuries. 137.

31v-41v “Beda Venerabilis”; 42v-135v “Idem, Computus sive de ratione
temporum.” Folios 22v-23r have illustrations of finger counting.
Catalogs


Found here are various listings of MSS with "algorism#".


Researched to find MS 644 suggested by Saxl and Meier's Catalogue. It does not reside in HMML's microfilm collection.


Various Computus MSS. found: MSS. 201, 202, 508, 509.


Used Vol. I (Class B) and Vol. III (Class O) also at HMML. Found listed references to "algorism#". I am unsure whether or not these MSS are at HMML.

Researched to find pp. 386-393 listing Ms Laud. Misc. 644 giving "tractatus algorismi" (Bl. 124r-125r). I found this manuscript then in the Bodleian Library Catalogue of H. O. Coxe. The MS is not in the HMML microfilm collection.


I used this source to find Wien MSS. used in research. HMML Catalogue.
Appendices
Appendix A: A Note on the Relativism in the Modern Perception of the Middle Ages

Dorothy V. Schrader, in “The Arithmetic of the Medieval Universities” in Swetz’s From Five Fingers to Infinity, states that:

“Although it is true that much of what was, in the medieval university, course material for a master’s degree is today common knowledge for third-grade school children, and although some of the more profound medieval processes of ratio and proportion are today taught in eight-grade arithmetic classes, medieval arithmetic must not be regarded as superficial or merely elementary. Many of the concepts are as challenging to modern graduate students of number theory as they were to medieval students of arithmetic. Moreover, the acceptance and spread of the Hindu-Arabic number system—a place system instead of an additive one, the use of zero as a number, number symbols instead of letter symbols, decimal instead of duodecimal fractions—with all of its implications and ramification, required the reeducation of the entire population of Europe, no small task to accomplish, even over two centuries” (238).

Frank J. Swetz states in From Five Fingers to Infinity that:

“The historical period from the close of the sixth century to approximately the beginning of the twelfth has frequently been referred to as the "Dark Ages" of European intellectual and cultural achievement. Such a designation results from a relative judgment based on retrospective comparison between the preceding accomplishments of the Greek world and those to come later during the Renaissance. True, chronologically, the "Dark Ages" existed between these two periods of illustrious human achievement, but, in itself, it was not a period of stagnation as has too easily been inferred. The centuries in question mark a period of strife, social experimentation, and societal and intellectual transition. The political void left by the fall of the Roman Empire was filled by the
institutional authority of the Catholic Church. Its priorities were spiritual not secular. Revelation was frequently placed in a position of official dominance over reason. While some priorities changed, the spirit of human ingenuity still thrived, and cultural and technological innovations abounded" (198).
HISTORICAL EXHIBIT 4.2

The Geometry of Gothic Church Windows

One of the outstanding architectural features of European Gothic churches is tracery, ornamental stone work of interlacing or branching arcs. Tracery depends on the ingenious and creative use of circular arcs and attests to the skill of medieval stonemasons as both craftsmen and users of geometry. Individual designs were quite simple but combined they resulted in structures of striking symmetry and beauty. Two tracery patterns used in constructing vaulted windows are given below.

Some other designs are:

Illustrations adapted from Renno Artmann, "The Clusters of Haurine," The Mathematical Intelligencer 13 (Spring 1991): 44–49, with the permission of Springer-Verlag and the cooperation of the author.
Appendix C: General Timeline

Dates: Western Europe and “Famous” Mathematicians

It may be important for the reader to place this discussion of numeric transition within a context already known. This timeline consists of some Western European events at the left with some corresponding dates of mathematicians on the right. Note that the top half of the page covers twice as many years as the lower half. This may help to express how some view medieval mathematics. This perception is the notion that little occurred in the realm of mathematics since there are few “famous” mathematicians to account for the period. The following is taken from p.70 of J. Fang’s Mathematicians: From Antiquity to Today.

<table>
<thead>
<tr>
<th>Year</th>
<th>Event</th>
<th>Year</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>636</td>
<td>Roman priest visits China</td>
<td>628</td>
<td>Brahmagupta</td>
</tr>
<tr>
<td>641</td>
<td>Library of Alexandria Burned</td>
<td>710</td>
<td>Venerable Bede</td>
</tr>
<tr>
<td>711</td>
<td>Saracens invade Spain</td>
<td>766</td>
<td>Brahmagupta in Arabic</td>
</tr>
<tr>
<td>771</td>
<td>Charlemagne reigns</td>
<td>775</td>
<td>Alcuin serves Charlemagne</td>
</tr>
<tr>
<td>790</td>
<td>Harun al-Rashid reigns</td>
<td>830</td>
<td>Al-Khowarizimi</td>
</tr>
<tr>
<td>850</td>
<td>Mahavira</td>
<td>900</td>
<td>Abu Kamil</td>
</tr>
<tr>
<td>871</td>
<td>Alfred the Great reigns</td>
<td>980</td>
<td>Abul Wefa</td>
</tr>
<tr>
<td>1000</td>
<td>Lief Ericson in Vinland</td>
<td>1000</td>
<td>Alhazen</td>
</tr>
<tr>
<td>1027</td>
<td>Mayas’ expansion</td>
<td></td>
<td>Pope Sylvester II</td>
</tr>
<tr>
<td>1066</td>
<td>Norman Conquest</td>
<td>1037</td>
<td>Avicenna dies</td>
</tr>
<tr>
<td>1076</td>
<td>Turks capture Jerusalem</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1095</td>
<td>First Crusade</td>
<td>1100</td>
<td>Omar Khayyam</td>
</tr>
<tr>
<td>1200</td>
<td>U. Paris Formed (Oxford, 1214; Padova, 1222; Napoli, 1224; Cambridge, 1231)</td>
<td>1202</td>
<td>Fibonacci</td>
</tr>
<tr>
<td>1241</td>
<td>Mongols invade Europe</td>
<td>1303</td>
<td>Chu Shih-chieh (Pascal’s Triangle)</td>
</tr>
<tr>
<td>1349</td>
<td>The Black Death</td>
<td>1328</td>
<td>Bradwardine</td>
</tr>
<tr>
<td>1431</td>
<td>Joan of Arc burned</td>
<td>1360</td>
<td>Oresme</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1450</td>
<td>Nicholas of Cusa</td>
</tr>
</tbody>
</table>
1456  Bible, printed with movable type
1478  First printed arithmetic
1482  First printed geometry
1492  Columbus discovers America
1517  Protestant Reformation
1521  Magellan discovers Philippines et. al.
1534  Loyola founds Jesuit order
1558  Elizabeth I reigns
1564  Shakespeare b.
1572  Massacre of St. Bartholomew
1579  Drake reaches California
1588  Spanish Armada destroyed
1598  Edict of Nantes

1460  v. Peurbach
1470  Regiomontanus
1484  Chuquet
1492  Pellos (Decimal point)
1500  Leonardo da Vinci
1510  Dürer
1518  Adam Riese
1520  Copernicus
1525  Rudolff
1540  Viète b.
1545  Cardano, Ferrari, Tartaglia
1564  Galilei b.
1572  Bombelli
1583  Clavius
1590  Cataldi; Steven
1593  Desargues b.
1596  Descartes b.
### Appendix D: Glossary

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS</td>
<td>Manuscript</td>
</tr>
<tr>
<td>MSS</td>
<td>Manuscripts</td>
</tr>
<tr>
<td>Folio</td>
<td>One page of a manuscript; considered both the front and the back</td>
</tr>
<tr>
<td>R.</td>
<td>Recto; as the MS is opened, this is the side of the folio on the reader’s right.</td>
</tr>
<tr>
<td>V.</td>
<td>Verso; as above, yet on the reader's left. This could be thought of as the &quot;back&quot; of the folio.</td>
</tr>
<tr>
<td>Algorism</td>
<td>Etymologically derived from al-Kwharizmi. This is the system of rules that when followed, will yield the result. For example, adding 23 and 81 follows a series of steps that will yield 104.</td>
</tr>
<tr>
<td>Calculator</td>
<td>The person doing the calculations (this is before electronics).</td>
</tr>
<tr>
<td>Multiplicand</td>
<td>The number of which will be multiplied.</td>
</tr>
<tr>
<td>Multiplier</td>
<td>The factor that gives how many times the multiplicand will be added.</td>
</tr>
<tr>
<td>Product</td>
<td>The result of the multiplication of the multiplicand as many times as indicated by the multiplier.</td>
</tr>
<tr>
<td>Dividend</td>
<td>The number to be divided.</td>
</tr>
<tr>
<td>Divisor</td>
<td>The factor that divides the dividend into the result.</td>
</tr>
<tr>
<td>Quotient</td>
<td>The result of how many times the divisor went into the dividend.</td>
</tr>
</tbody>
</table>
Figure 28: Boethius Finger-Counting

"Many mediaeval miniatures and tapestries show us scholars, usually astronomers, counting on their fingers, which are bent in various ways, and this is how the fifteenth-century painter, [Justus van Ghent], showed Boethius . . . which was painted for the Ducal Palace at Urbino, where it may still be seen" (Dedron, 166)
Figure 29: Bede’s Finger Mathematics

“Bede was a British monk who lived in the seventh century (ca. 673-735) and who was considered to be one of the greatest medieval Church scholars. Among his works were mathematical tracts on the Church calendar, i.e. computi, ancient number theory, and finger mathematics or numeration which was actually a method for designating numbers by the use of finger gestures. The . . . diagram, published a thousand years after Bede’s death, illustrates some of Bede’s medieval number postures” (Swetz, Fingers, 220)
Figure 30: Finger Reckoning

Gestures representing the numbers 30-90 (in tens) and 100-500 (in hundreds). Note the change of hand at 100. Extract from a manuscript of the De temporum ratione by Bede, copied in Italy at the beginning of the fourteenth century.
National Library (Taton xv)
Figure 31: The Art of Finger Reckoning
Figure 31: The Art of Finger Reckoning

“One scheme for finger reckoning was developed into a fine art during this period. In fact, a citizen who could not handle such a system competently was at a real disadvantage...”(Linn,105)

Figure 32: Finger Symbolism in the 13th Century

“From the Codex Alcobatiensis in the Biblioteca Nacional at Madrid, dating from c. 1200. From a photograph by Professor J.M. Burnam”(Smith, History II, 198).
Figure 33: Pacioli on Finger Symbolism

"[This image is from] the *Suma* of Pacioli, Venice, 1494. The two columns at the left represent the left hand, the other two represent the right hand" (Smith, History II, 199). This image shows how numbers less than one hundred were counted on the left hand and those greater than one hundred on the right. Juvenal refers to this custom in his tenth satire, saying: "Happy is he indeed who
has postponed the hour of his death so long and finally numbers his years upon
his right hand” (Smith, History II, 197). 68

---

68This citation was noted on p.359 in Philip S. Jones’ “Tangible Arithmetic IV: Finger Reckoning and Other Devices” (359-363) in Swetz, *From Five Fingers to Infinity*. 
Appendix F: Examples of Abacus

Figure 34: Chinese Abacus

"On this Chinese abacus, of unknown date, the upper rank of counters represents five and the lower rank represents units. The figure shown, from left to right, are 0, 0, 7, 2, 3, 0, 1, 8, 9" (McLeish, NP, center section).

Next Page:
Figure 35: The Salamis Abacus

"While there is some question as to the figure on the Darius vase, there seems to be little respecting an abacus found on the island of Salamis. It is of white marble, 1.49 m. long and .75 m. wide, and is broken into two unequal parts, but is otherwise well preserved and is now in the Epigraphical Museum at Athens. . . . It may have been the computing table in the counting house of some dealer in exchange, and in some of its features it is not unlike the tables used by bankers in the Middle Ages; or, . . . it may have been used in some school. The theory that it may have been used in scoring games of some kind seems to have no substantial foundation. In any case it was apparently used for the mechanical representation of numbers by means of counters" (Smith, History II, 162).
Appendix G: Explanation of Abacus Calculations

The following is as an explanatory narrative on the abacus from Michael S. Mahoney's, "Mathematics" in Lindberg, *Science in the Middle Ages*.

"In both the smaller hand format and the larger board form, [the abacus] consisted of several columns containing successive powers of 10 counted from the right and beginning with 1. The particular digit within each column was represented by small stones (*calculi*, whence *calculare*, to "calculate"), each counting as 1 or, in some versions, as 5 when placed above a horizontal line drawn through the columns. Addition required nothing more than laying down the numbers and regrouping columns in which the total exceeded 9; subtraction followed an equally simple inverse pattern. Multiplication demanded two supplementary steps: first, calculation of the individual, digit-by-digit subproducts (by continued addition on the board or by reference to a 9X9 multiplication table or even by finger-reckoning); second, determination of the proper column into which to place the product of two other columns (for example, 10s columns X 100s column = 1000s column). Once laid down, the subproducts directly yielded the final result, needing regrouping at most. For example, 496 X 23 = 11,408 followed from the sequence of operations presented in [the figure below]. Division was only slightly more complicated and followed two basic patterns: "golden division" computed directly with the divisor, while "iron division" supplemented the divisor to the next multiple of 10 or 100, adding the product of subquotient and supplement to the remainder before each successive subdivision"(147-8).
Figure 36: Diagram of Abacus: 496 x 23
Figure 37: Examples in the History of the Counting Board

"Various forms taken by the counting board throughout history; the number 2704 is represented on all of them" (Menninger, 372).
"The Darius Vase with its depiction of a calculator at his reckoning table (bottom row), one of only two known ancient representations. Height 1.3 m., greatest circumference 2 m. Probably 4th century B.C. Museo Nationale Naples" (Menninger 303).
Figure 39: "The Treasurer on the Darius Vase, seated at his reckoning table, which has counters on it" (Menninger 304).
Figure 40: Arithmetic Personified

“Arithmetic personified, instructing noble pupils in the methods of computation with counters...The inscription embroidered at the bottom is in praise of the art of numbers. French Tapestry about 3 x 3 m. Musée de Cluny, Paris” (Menninger 365).
Figure 41: Reckoners at Work

"A view of a 16th-century German counting house, in which book-keepers and reckoners are at work. Woodcut by Jost Ammann" (Menninger, 433).
"One of the three known German reckoning tables in existence, from Dinkelsbühl (now in the Museum there). Both counting areas are carved into the surface of the table; each has two areas for computations. Photographed by P. Hammerich" (Menninger, 344).
Figure 44: Discussion Over a Counting Table

"Counting table with specified coin rows (called a coin-board or number-table). This woodcut probably from Strasbourg" (Menninger, 340)
Appendix J: An Explanation of Table Reckoning

Vera Sanford: "Counters: Computing if You Can Count to Five" in Swetz's From Five Fingers to Infinity. The following is an explanation of calculations done with the loose counter abacus.

"The loose counter abacus of the sixteenth century differed from the Greek one in that the lines were marked from right to left instead of up and down. The line nearest to the computer had the lowest value. The counting board might be of stone with the lines cut on it. It might be of wood with lines drawn in chalk for temporary use or painted on permanently. It might be a table cloth with the lines embroidered on it. The lines had the values 1, 10, 100, etc., and the spaces between had the values 5, 50, 500. There is a close correspondence between these markings and Roman numerals. A number is indicated by placing counters on the lines and spaces as the case requires. The accompanying figure shows how the numbers would appear.

Figure 45: 1285 and 431 make 1716

"In addition, the addends are indicated on the counting board. Then whenever a line has five counters on it, as is the case with the tens line and the hundreds line in the case given here, the five counters are picked up and one is "carried" to the next space. In the example under consideration, there now are two counters in the fifties space. But two fifties make one hundred, so these two counters are picked up and a counter is laid on the hundreds line. The process is repeated until no line has more than four counters and no space more than one.
"In subtraction, minuend and subtrahend are entered on the counting board. Then the counters of the subtrahend are matched with those of the minuend and each pair is removed from the board. In some cases it is necessary to "borrow" a counter of higher value from the minuend and replace it by the equivalent value of counters in the next lower space or line. The process continues until no counters are left in the subtrahend.

"To multiply a number by 10, the counters are laid out as if the tens line were actually the units line. To multiply by 100, the hundreds line represents the units. To multiply by 200, you multiply by 100 twice. Since 50 is half of 100, multiplying by 50 is accomplished by taking half of the number of counters on each line or space in 100 times the number. In the following example the number 284 is multiplied by 153. (See solution below.)

![Diagram of counters]

Figure 46: 284 times 153 is 43,462

"Division is difficult and a number of different methods are used. The computer is expected to know the multiplication facts. He decides on the proper quotient figure, and subtracts the partial product from the dividend, using the various lines as the units line as was done in multiplication.

"Except for the process of division, computation with the loose counter abacus made no demands on the learner beyond learning how to enter the counters, how to read a number represented by counters, and how to count to five. Multiplication was clumsy. Division demanded a knowledge of multiplication combinations unless the computer avoided this issue by using repeated subtraction."
"The loose counter abacus is simpler and slower than is the Chinese or the Japanese abacus which requires more mental work. On the other hand, it is a simpler device and one which is easier to master" (222-223).
Appendix K: A Note of Comparison Between the Roman and Hindu-Arabic Numerals

The following is taken from Al-Daffá's *The Muslim Contribution to Mathematics*:

"In actual fact Europe adopted [Arabic numerals] from the Muslims only in the thirteenth century. Fighting their introduction and that of the decimal system that went with them for several hundred years, Europe deprived itself of the advantages of one of the world's greatest contributions to mathematics.

"Prior to the Arabian numerals, the West relied upon the clumsy system of Roman numerals, and before that upon even more clumsy Greek numerals. In the decimal system, the number 1843 can be written in four numerals, whereas in the Roman numerals, eleven figures are needed. The result is MDCCCXLIII. It is obvious that even for the result of the simplest arithmetical problem, Roman numerals called for an enormous expenditure of time and labor. The Arabian numerals, on the other hand, render even complicated mathematical tasks relatively simple. Professor J. Houston Banks of Peabody College has stated:

"The Roman system seems to have some advantage over the present numerals when we consider the process of addition. Let us consider the addition of 127 and 58 in Roman numerals:

CXXVII

LVIII

CLXXXVIIIIII

"We can merely bring down all the symbols in each addend; then if we remember that five I's are written a V and two V's as X, the answer becomes CLXXXV. It is not necessary to know the addition combination such a 7 + 8 and 5 + 2. However, the process is much longer and more cumbersome. But if we attempt to multiply or divide, it is a different story" (33-34).
Appendix L: Examples of the Physical Transition that Occurred to the Hindu-Arabic Numerals Over the Centuries

Figure 47: The Family Tree of the Indian Numerals
Figure 48: Table for the History of Hindu-Arabic Numerals

This table shows how the physical representation of the numerals changed over the centuries. A great deal of research has been done in this specific area, especially by Menninger.
Appendix M: Al-Kwarismi: Lexicon

The asterisked entries appeared most often in my research. “Al-Khwarizmi” was used most often in recent scholarship.

Abu Jafar Muhammad ibn Musa al-Khwarizmi (Menninger 410)
Mohammed ibn Musa al-Khowarizmi (Rogers 67)
*Al-Khwarizmi (Al-Daffá 33)
(McLeish 139)
(Swetz, Capitalism, 27)

*Al-Khowarizmi (Cajori, History of Mathematics, 118)
(Bell 93)

Al-Khowârizmî (Rogers 67)
Al-Khwarizmi (Smith, History II, 72)
Levey ix)
Al-Khowârizmî (Cajori, Notation Vol.I)
Al-Khovarizmi (Dedron 206)
Alchwarizmi (Al-Daffá 33)
Al-Karismi (Al-Daffá 33)
Al-Hwarismi (Grabois 37)
Algoritmi (Al-Daffá 33)
Algoritmi (Swetz, Capitalism, 28)
Algorithmus (Swetz, Capitalism, 28)
Algorismus (Menninger 412)
Algorismi (Al-Daffá 33)
Algorysine (Old English) (Sanford, Swetz 221)
Al Gore is Me (MJS)
Appendix N: A Brief Example of Algorist Multiplication

The following text on algorist division is taken from a note in Mahoney’s chapter “Mathematics” in David C. Lindberg’s *Science in the Middle Ages*.

"Early algorists appear to have worked on sand or wax, drawing symbols where they once dropped stones into columns. Erasure on such surfaces was easy and, hence, emulation of the techniques of the abacus followed naturally. On parchment or paper, using ink, it was less simple to eradicate symbols; so one crossed out, often trying to array the intermediate results in a pleasing pattern, as in the "sail" for multiplication (using the example given below):“ (172).

```
4
3
20

1109
9278
8496
50
23
23

496 x 23 = 11408
```

Figure 49: 496 times 23 makes 11408, Algorist Multiplication method
### Appendix O: Examples of Algorist Division

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>65284</td>
<td>I</td>
<td>16</td>
</tr>
<tr>
<td>594</td>
<td>66284</td>
<td>66284</td>
</tr>
<tr>
<td></td>
<td>694</td>
<td>694</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>168</td>
<td>5</td>
</tr>
<tr>
<td>168</td>
<td>66284</td>
<td>168</td>
</tr>
<tr>
<td>694</td>
<td>694</td>
<td>694</td>
</tr>
</tbody>
</table>

Figure 50: Galley Method of Algorist Division

<table>
<thead>
<tr>
<th>(15)</th>
<th>(109)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td></td>
</tr>
<tr>
<td>16878</td>
<td></td>
</tr>
<tr>
<td>66284</td>
<td></td>
</tr>
<tr>
<td>69444</td>
<td></td>
</tr>
<tr>
<td>699</td>
<td></td>
</tr>
</tbody>
</table>

Figure 51: Galley Method of Algorist Division
Figure 52: Strike-out Division: 7825 / 43

Figure 53: Strike-out Division in a Text

"A "Strike-out" division, from a 15th-century manuscript... Bayrische Staatsbibliothek, Munich" (Menninger, 331).
Figure 54: Interpretive Galley Division (from a monk with too much time on his hands)

"A sixteenth-century illustration of galley division done by a Venetian monk" (Swetz, Capitalism, 215).
Appendix P: Selection from *Libro de moralidades*, edited by George Greenia

The *Libro de moralidades*

This long didactic work is found in a manuscript now at the Biblioteca Nacional in Madrid. Formerly part of the holdings of the Biblioteca del Cabildo (de la Catedral) de Toledo (MS 98-2), it is now part of the general BN manuscript holdings under the number 10011.

The first part of the volume is devoted to a treatise in Catalan on surgery (ff. f-34r). As a scientific work apparently translated originally from the Arabic, it was studied by Millàs Vallicrosa in his *Traducciones orientales* (Madrid: Consejo Superior de Investigaciones Científicas, 1945).

Folios 35a-75a are in a different hand, and are considered together in the third edition of *Bibliography of Old Spanish Texts* (Madison, 1984) as simply *Libro de moralidades* (#1641), but were analyzed in detail again by Millàs who apparently coined this title for his 1945 catalogue. The first half of this middle section is a conflation of many sources, again mostly translated from the Arabic, and leaning heavily on the work of Al-Ghazali (al-Ghazzali = Abu Khamad Muhammad Ibn Muhammad Altusi Alsha'i, 1058-1111). The second half is the text is a letter addressed to Alexander by his father. Millàs also provided a critical description (137-39) and a transcription (340-48) of the final portion of the manuscript, ff. 74b-77b, a version of the learned dialogue entitled "Diálogo entre Perticus y el duque Adriano".

* * *

Biblioteca Nacional, Madrid MS 10011 (olim Toledo, Biblioteca de la catedral 98-2) is written on thick paper without watermarks, 328 x 255 mm, 2 cols., each 255 x 80 mm, 43 lines high; blank spaces (15-25 (=3-5 lines) x 10 mm) have been left for initials, and more blank space between paragraphs (15 mm), perhaps for rubrics, none of which were never completed; there is very light vertical ruling only. The hand was dated by Millàs as around the XIV-XV centuries.

The text, occupying ff. 35a-75a, is acephalous, but only a single page is missing; the folios are number in pen in a different but roughly contemporary hand from 2 to 47 at the bottom right of each verso starting with 35r. Below these numbers there are catch words as well on most folios. The MS starts with three blanks folios and ends with four more that because of the continuous worm holes must belong to the original collation. There are also two folios at the beginning and end that are in better physical condition but perhaps of the same period.

The numbering on the folios is deficient: the first four blank folios are numbered although the first text does not start until f. 5r. There is an unnumbered folio between ff. 26 and 27 and two folios numbered 51 and two more numbered 74. I have corrected the numbering system to conform with the norms of transcription of the Hispanic Seminary

---

1 Domínguez Bordona, 1931, 73-4.
giving present foliation marks as well. A description of the organization by quires is unfortunately impossible because in the last (nineteenth-century?) binding of this codex many of the sheets were individually tipped onto tabs to provide a stronger sewing surface, and it is even likely, given the presence of the contemporary catch words on every sheet, that this assembly was copied onto loose sheets from the outset.

I transcribe the text with a minimum of modernization, employing some punctuation, capitalization and accent marks, but leaving the copyist's slash marks (simple slash (/), simple dot (·), and slash with dot (/·)) in the manuscript in their place to indicate his intended pauses, although his usage was not consistent and sometimes they are used as mere line fillers. No distinction is made here between the different forms of "s" (s, s, i), but "z" and "v" are transcribed as such whenever present; "i" (i or j in the text) is distinguished from "y"; "b", "v" and "u" are kept as written; and initial trilled "r" (rendered as "R" in the MS) transcribed as simple "r".

All abbreviations that are resolved into full forms are indicated by italics. Words joined together are generally transcribed as written. Written accents are used only to distinguish homographs, such as "vos/vos", "el/él", "mas/más", and periodic cases of apocope of enclitics flagged for the reader with apostrophes. Words or letters to be deleted are printed between curved brackets, while words or letters which need to be supplied are printed between pointed braces (<>); pointed brackets with an asterisk (<*>) represent characters lost because of occasional worm holes in the paper. The paragraph divisions below are those of the manuscript.

MS BN 13037 (= Burriel catalogue of Toledo collection; signatura no longer valid), uncited by Millés in Traducciones, f. 27:

343 Anonymo. Un cuerpo, en q. estan 5 tratados de la Cirugia en lengua Catalana. Item. se siguen dos tratados, Pr.o de virtudes y vicios, y otro de la historia natural de caelo, Meteoreos, Geographia, Animales de y a lo ultimo, un dialogo moral, en Castellano. Papel. letra de 1400. .22.26.fol.
Arithmetic teaches how to count and has seven manners. The first to adjust the least tally with the greatest, such as -7 - with -8 - all together make -15 - The second shows how to take the lesser tally from the greater, which is from xxv to take x there remains xv. The one [sic; 3rd] shows how to double, like -v - and -v - make ten. The 4th shows how to split in half, so that ten times -v - makes fifty. The 5th shows how to multiple so that as with -x - times x que make C. And the 6th shows how to divide the equal portions in tallies such as when one wishes to divide -C - by -9 -1 - horses that [one] taken [away] comes to -x -. And the 7th shows + (blank) such as -v - times -v - make -xxv -, from -xxxv - is -v -. And in this seventh way there are 2 ways and the first is by squared tallies and the second is by + (blank).

The squared tally = (blank) = times like -v - times -v - makes -xxv - and the 2 times three makes Nine + (blank).

The blank is the tally, -v - dos times xxv make -c xxv - and this art is counted with x - figures which are these that follow: -1 -2 -3 -4 -5 -6 7 8 9 0 and the first of these letters means -one and the second two so on the following ones until -x -
ENDNOTES

1. This is the most vexed section of the text, partly because of the confused arithmetical examples and partly because he author (or copyist) switches back and forth between Arabic and Roman numerals. The four blanks in this short segment of the MS suggest much doubt.

2. Here the copyist uses pointers like inverted Vs to indicate where he was to insert text latter.
Codex Vindobonensis Palatinus 10585
Wien. Österreichische Nationalbibliothek. HMM # 19,556

[folio 118]
Uno tiene -4- diferencias de lanas conviene a saber
Una suerte que vale  @ -12- [$] y otra -21- y otra
23- y otra a -27- quiere mezclar todas y saber
200- @ que valga cada @ a -19- [$] demando que cantidad se tomará de cada suerte.

[calculations]

Les <h>a de tomar de lo que valía 12 [$]. 83-@ 12
36

delas que valía 21 – 38 . 32
36

delas que valía 24 – otros – 38 32
36

delas que valía 27 – otros – 38 32
36

200 108 36

* * *

There are -4- differences of wool, that is to say
one kind that is worth [?] at -12- [$] and another [at] -21- and
another
23- and another at -27- one may wish to mix {combine?} them all and
that would be
200- @ what would be each worth -19- [$] ??
?? what amount might be taken in each case.

[calculations]

One should take from them what's worth
12 [$]. 83 – 12
36

from those worth 21 – 38 . 32
36

from those worth 24 – others– 38 32
36

from those worth 27 – others– 38 32
36

200 108 36
Figure 55: Computational Dispute

"'Abacist' and 'algorithmicist' dispute the advantages of computations on the counting board and with place-value numerals. The man at the rear works on paper with ink and pen, the one at front right with chalk. The letters on the wall stand for *verbum domini manet in eternum*, "God's Word remains forever." From an English work on the seven liberal arts by Robert Recorde, personal physician to the King" (Menninger, 432).
Appendix S: A Note on the Treviso Arithmetic and Similar Mathematical Manuscripts

The following is taken from the preface of Frank J. Swetz's work, Capitalism and Arithmetic.

"In the ensuing months after my initial encounter with Smith's papers, my thoughts repeatedly returned to the Treviso translation. Gradually, its importance and significance dawned upon me. The fifteenth century was a time of commercial, intellectual, and mathematical ferment, yet there is a dearth of specific information on the mathematical climate of this period. Some work has been done in this area, notably by Gino Arrighi of Italy, Kurt Vogel of Germany, and, most recently, by Warren Van Egmond of the United States. Arrighi, perhaps the most active contemporary researcher on the history of mathematics in medieval and early Renaissance Italy, has produced a rather extensive collection of research publications that reflect on this subject. Vogel, of the Deutsches Museum, Munich, has published two major works that contribute to an understanding of the mathematical activity of this time: Die Practica des Algorismus Ratisbonensis: Ein Rechenbuch des Benediktiner klosters St. Emmeran aus der Mitte des 15 Jahrhunderts (1954), and Ein italienisches Rechenbuch aus dem 14 Jahrhundert (1977). However, the works of both these men are limited to readers of their respective native languages, Italian and German, and are thus not readily available to a wide transatlantic audience. Dr. Van Egmond's scholarly study, The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300-1500 (1976), focuses on the broad impact of "abaci" manuscripts on mathematical thinking, and attempts to view their contents and development within an economic and sociological framework. Unfortunately, its format and availability as a doctoral dissertation limits its accessibility and appeal to a general reading audience. Thus, I began to realize that Smith's work was the first and only English language translation of a fifteenth-century European arithmetic book that I had encountered. Its contents, if analyzed, could provide valuable information. A little further research revealed that the Treviso manuscript was, indeed, an intellectual and historical treasure for several reasons:
1. It is the earliest known dated, printed arithmetic book.

2. It is one of the first mathematics books written for popular consumption and, as such, marks a turning point in the history of human knowledge. The book's message, transcribed in the common Venetian dialect of the period, was intended for all who wished to learn the art of computation, not just for a privileged few, as had been the case previously.

3. The contents of the work provide an early example of efforts to promote the Hindu-Arabic numeral system and its computational algorithms. Use of the abacus and Roman numerals was still popular in much of Europe, although to a lesser degree in Italy, where commercial interests demanded more efficient computational procedures. Thus, the translation supplies insights into the mathematical climate and controversy of the late fifteenth century.

"More fully aware of the importance of the manuscript I had uncovered, I undertook to complete the task that David Eugene Smith had begun nearly eighty years before—to publish a study of early Renaissance arithmetic based on the contents of the 1478 Treviso tome. Smith's concern was mainly pedagogical; he was primarily interested in how early arithmetic was taught. The following study, while considering the heuristics of early European arithmetic instruction, will also investigate the mathematical and sociological significance of the Treviso Arithmetic's contents.

"The first chapter of this study provides a general background in which the Treviso Arithmetic and its contents must be understood. Chapter Two presents Smith's free translation of the Arithmetic. The succeeding four chapters serve as a commentary on its contents and correspond to the sequential ordering of topics in the Arithmetic: numeration, addition and subtraction, multiplication, and division, and the techniques of problem solving. Chapter Seven focuses on the social, economic and commercial aspects of the book's contents, and summarizes the findings of this excursion into early Renaissance mathematics.

"Ostensibly, this study focuses on a book and its contents but, perhaps more importantly, it also concerns a time (the early Renaissance), a place (the Venetian Republic) and circumstances (the rise of mercantile capitalism and the
economic beginnings of industrialization), and how these three aspects molded and affected the directions of human involvement with mathematics. Ultimately, the object of this work is to contribute to a better understanding of the historical interaction of the development of mathematical ideas and techniques and their societal milieu” (xiv-xvii).
Frank J. Swetz, in his work: *Capitalism and Arithmetic*, states that:

“While the information and computational techniques considered in Fibonacci’s *Liber abaci* appealed enough to merchants to ensure its preservation and transmission in their practicae as *abaci*, its adoption by European society, even among the merchant class, was not rapid. At first, the new knowledge seemed to have as great impact, but its spread in the thirteenth and fourteenth centuries was hesitant. Even in the fifteenth century, Pacioli lamented those merchants who still used "old and vulgar methods" in their arithmetic. However, it was at the end of this same century, in the period of the *Treviso’s* publication, that the momentum for using the new arithmetic increased greatly. For the merchant class, the late Middle Ages—the initial period in which they were exposed to the Hindu-Arabic numerals—was a time of extreme commercial opportunity and easy profit. As such, there may have seemed no need for the introduction of radical changes in the notations of record keeping and the techniques of computation. In some cases, vested interests further resisted reforms.

“Gradually, a series of events changed the mercantile climate and forced a rethinking of business procedures. The collapse of the Mongol Empire and the rise of the Ottomans curtailed much Eastern trade. The Black Death and the ravages of the Hundred Years War reduced European commerce so that, by the fifteenth century, Europe was experiencing economic depression. In this time of shrinking markets, smaller profits, and keener competition, merchants re-evaluated their situation and sought out more rational methods of operation. Thus merchants were once again attracted to the efficiency offered by the use of the new arithmetic.” (293-294).
Appendix U: A Note on Society and Mathematics

Frank J. Swetz offers some final thoughts in Capitalism and Arithmetic.

"This brief consideration has exposed a link in the chain of historical transmission of mathematical knowledge and aids in the attainment of a better understanding of how mathematics and society are interrelated. While the concepts and techniques provided by the Hindu-Arabic numeral system were known in Europe for five hundred years prior to the appearance of the Treviso Arithmetic, their full development and exploitation lay dormant until a favorable social climate existed in which they could be appreciated. This climate existed in 14-15th century Italy, particularly in Venice, where the spirit of boldness that was so much a part of early Venetian survival now turned to financial adventurism and the building of a commercial empire. Traders became international merchants, and the growth and reinvestment of wealth saw the rise to commercial capitalism and its associated institutions—a system of international monetary exchange based on the ducat, corporations and stock companies, and a system of deposit banking. As geographical and economic constraints were challenged and overcome, new intellectual and political horizons were also explored. Secularism and political independence from Rome provided a climate of freedom where unfamiliar knowledge of subjects such as arithmetic and mathematics was consciously pursued. In this period, a modern realization was dawning as to the usefulness and facility of arithmetic; these benefits gave mathematics a new value. While this vitalized appreciation first centered on arithmetic, for similar reasons it would soon spread to other areas of mathematical endeavor: geometry, trigonometry and algebra, and other sciences. (295-296).