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# Three-Player GEN on Groups

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**Abstract**

The three-player game of GEN involves taking turns selecting elements of a group to add to a common pool of elements. Each element can only be selected once, and all players share the pool of elements. The object of the game is to be the player that adds the final element to the pool that will generate the entire group. We will identify and develop the strategies players must follow in order to win the game when playing with cyclic groups, dihedral groups, and nilpotent groups.

# Chapter 1

# Introduction

The games of Do Generate (GEN) and Do Not Generate were first discussed by Anderson and Harary[1]. They examined both of the games as two-player scenarios in which players would either attempt to generate, or not generate, a group $G$ by adding elements of $G$ to a common set $S$. The game would end when $\langle S \rangle = G$. In GEN, the winner of the game is the player that adds the final element to $S$ so that $\langle S \rangle = G$. In Do Not Generate, the loser of the game is the one that selects an element to add to $S$ such that $\langle S \rangle = G$.

Variations of Do Not Generate have been examined further by Benesh, Ernst, and Sieben[2], and also by Benesh and Gaetz[3]. In particular, Benesh and Gaetz examined the three-player avoidance game of Do Not Generate. The extra player within Anderson and Harary's original game led to some interesting differences in the winning strategies of the two-player and three-player games.

While Benesh and Gaetz examined three-player Do Not Generate, there has not been any research done on the three-player game of GEN. In this paper, we will attempt to try identifying the winning scenarios on different groups for each player in the game.

Before we dive into the different scenarios that the game of Three-Player GEN can take on, it is important that we brush up on a few definitions that are essential for understanding the game. These terms and definitions will be referenced throughout the entire paper.

## 1.1 Definitions

### Groups

In mathematics, a *group* $(G, *)$ is a set of elements paired with a binary operation $*$ that satisfies the following requirements:

1. For all $a, b, c \in G$, the operation $*$ is *associative*. That is:

$$(a * b) * c = a * (b * c)$$

2. There exists a special element $e \in G$, called the *identity*, that follows the property:

$$a * e = a = e * a$$

3. For every element $a \in G$, there exists an *inverse* $a^{-1} \in G$ such that:

$$a * a^{-1} = e = a^{-1} * a$$

4. Finally, elements of $G$ will be *closed* under the operation. That is, for any $a, b \in G$:

$$a * b \in G$$

These requirements form the definition of a group, as described by Judson[6,40]. The integers under addition, real numbers under addition, and symmetries of a triangle under composition are all examples of groups. For the following few definitions, think of our group $G$ as $\mathbb{Z}_{12}$, the integers (mod 12) under addition.

### Subsets

The three-player game of GEN involves taking turns selecting elements of a group to add to a common pool of elements. Each element can only be selected once, and all players share the pool of elements. The object of the game is to be the player who adds an element to the pool so that the elements in the pool will generate the entire group.

This "pool" of elements is mathematically known as a subset of the group that we are using for the game. A subset $S$ is a collection of any number of elements from our group $G$. For example, $S$ could be $\{2, 5, 6\}$ with $G = \mathbb{Z}_{12}$. A subset may or may not have the same properties as a group. This depends on which elements are contained within the subset. If the elements within the subset satisfy all of the requirements of a group, then it is called a subgroup.

**Subgroups**

A set of elements $S$ is called a *subgroup* of a group $G$, $S \subseteq G$, if it is the case that for every $x \in S$, we have that $x \in G$, and $S$ has all four of the properties that a group has. That is, $S$ is a group when examined independently from $G$. A subgroup of our group $G = \mathbb{Z}_{12}$ is $\{0, 3, 6, 9\}$. The subgroup has all of the properties that are required in order to be a group. Clearly, all of the elements in the subgroup belong to the larger group, $G$.

For the three-player game of GEN, we will need to consider what the elements in our subset $S$ can generate. In order to be generated, an element must be produced by the finite composition of elements that are already contained in $S$ under the binary operation of $G$.

**Proper Subgroup**

A *proper subgroup* of a group, denoted $P \subset G$ is a subgroup that contains strictly fewer elements than are found in $G$. That is, all $x \in P$ are such that $x \in G$, but there is at least one element $y \in G$ such that $y \notin P$. $\{0, 3, 6, 9\}$ is an example of a proper subgroup. Technically, $G$ is a subgroup of itself, but it is not a proper subgroup of itself.

**Cyclic Subgroups**

For the purposes of our game (especially for one of the groups we will examine), it is important to understand *cyclic subgroups*. The following excerpt from Judson[6,56] provides us with a great definition of cyclic subgroups. While this definition will be essential in understanding the rather simple game of Three-Player GEN on cyclic groups, it also helps us begin to grasp the idea of subgroup generation for the later families of groups.

**Definition 1.1** (Judson)**.** *Let $G$ be a group and $a$ be any element in $G$. Then the set*

$$\langle a \rangle = \{a^k : k \in \mathbb{Z}\}$$

*is a subgroup of $G$.*

Judson goes on to prove that $\langle a \rangle$ is the smallest subgroup of $G$ that contains $a$. For our group $G$, we can take an element like $\{9\}$ and produce a cyclic subgroup of $\langle\{9\}\rangle = \{0, 3, 6, 9\}$. Some elements generate the entire group $G$, for some groups. Take $\langle\{5\}\rangle$ for example. We have that $\langle\{5\}\rangle = G$, the entire group. So $\langle\{5\}\rangle$ is not a proper subgroup.

As was similarly explained by Judson, we have that for any set of elements $X = \{x_1, x_2, ...x_m\} \subset G$, $\langle X \rangle$ is the smallest subgroup of $G$ containing $X$.

### Maximal Subgroups

A *maximal subgroup* $M$ is a proper subgroup of a group $G$ such that $M$ is not contained in any other proper subgroup of $G$. That is, there does not exist a proper subgroup $N \subset G$ that contains all elements of $M$ and additional elements. For example, $S = \{0, 4, 8\}$ is a subgroup of $G$, but it is not maximal. We have a subgroup $M = \{0, 2, 4, 6, 8, 10\}$ that contains all of the elements of $S$ and more. However, there is no proper subgroup that contains $M$ and additional elements of $G$. So $M$ is maximal.

### Frattini Subgroups

The *Frattini subgroup* of a group $G$, denoted $\Phi(G)$, is the subgroup composed of the elements found in the intersection of all maximal subgroups. That is, for an element $x$ to be in the Frattini subgroup, it must be the case that for any maximal subgroup $M$, $x \in M$. The maximal subgroups for $G = \mathbb{Z}_{12}$ are $M_1 = \{0, 2, 4, 6, 8, 10\}$ and $M_2 = \{0, 3, 6, 9\}$. Thus, we have that $\Phi(G) = M_1 \cap M_2 = \{0, 6\}$.

While not honed in on right away, Frattini subgroups will be essential in the development of winning strategies for the game on nilpotent groups. From Theorem 2 of Dlab[4], we have that for groups $A$ and $B$, it is the case that $\Phi(A) \times \Phi(B) = \Phi(A \times B)$. That is, the direct product of two Frattini subgroups is equal to the Frattini of the direct product of the two groups.

**Safe Set**

For the purposes of the game of Three-Player GEN, we will often talk about the "safe" elements of a group. This set of "safe" elements, denoted as $H$, is a set $H \subseteq G$ such that for all $x \in G$, $\langle H, x \rangle < G$.

## 1.2 Basic Rules of the Game

To begin a game of Three-Player GEN, we first need to determine which group $G$ will be used. Once $G$ has been identified, players will begin to add elements of $G$ to a shared set of elements. To begin, let this set $S_0 = \emptyset$. Once the elements in $S_n$ are able to generate all of $G$, the game is over. Three-Player GEN is played by the following rules:

1. Player 1 begins by selecting an element $g_1$ from $G$ and adding it to $S_0$. Now, we have $S_1 = \langle g_1 \rangle$.
2. If $\langle S_1 \rangle$ generates $G$, Player 1 wins.
3. If $\langle S_1 \rangle$ is unable to generate $G$, Player 2 selects a new element $g_2$ from $G$ to add to $S_1$. Now, we have $S_2 = \langle g_1, g_2 \rangle$.
4. If $\langle S_2 \rangle$ generates $G$, Player 2 wins.
5. If $\langle S_2 \rangle$ is unable to generate $G$, Player 3 selects a new element $g_3$ from $G$ to add to $S_2$. We then have $S_3 = \langle g_1, g_2, g_3 \rangle$.
6. If $\langle S_3 \rangle$ generates $G$, Player 3 wins.
7. If $\langle S_3 \rangle$ is unable to generate $G$, the game continues.
8. Players take turns adding elements to $S_n$ until there is a winner.

For our rendition of the game, it will be the case that if a player does not have a path to victory, they will attempt to help the person before them win. For instance, Player 2 would rather have Player 1 win than Player 3. The game is over when the elements within $S$ are able to completely generate $G$. The player who is the last one to add an element to $S$ is pronounced the winner.

Note: For the purposes of this paper, our clarification following the rules states that a player would rather have the person before them win than the person after. This is an alteration to our rendition of the three-player game that can be different elsewhere. For instance, players could play the game with the rule that they would rather have the following person win that the preceding person. They could also play a game in which there are six possible outcomes. Either Player 1, 2, or 3 wins outright, or one of the players determines which of the other two will win. These are variations of the game that can be studied in the future.

# Chapter 2

# Cyclic Groups

## 2.1 Introduction

As has been defined earlier, a cyclic subgroup, $\langle a \rangle$, is the subgroup of elements that are generated solely by the element $a$. In order for a group $G$ to be cyclic, it must be the case that $\langle g \rangle = G$ for at least one $g \in G$. Any element $g$ that has this property is known as a generator of the group.

Example of cyclic groups include the integers modulo $n$ under addition, such as $\mathbb{Z}_2$ or $\mathbb{Z}_{10}$. Clearly, we can have multiple elements of each group that, when composed with themselves repeatedly, generate the entire group. For $\mathbb{Z}_2$, the generator is 1 and for $\mathbb{Z}_{10}$, the generators are $1, 3, 7, 9$.

## 2.2 Three-Player GEN with Cyclic Groups

As outlined in Section 2.1, cyclic groups contain one or more elements that are defined as generators. These generators prove to be quite the unfair advantage for Player 1 during a game of Three-Player GEN.

**Theorem 2.1.** *Player 1 has the winning strategy of Three-Player GEN with a cyclic group.*

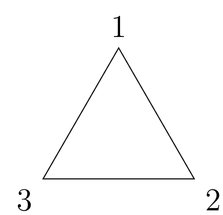
*Proof.* By the definition of a cyclic group, there exists at least one generator $g \in G$ such that $\langle g \rangle = G$. Thus, when Player 1 begins the game, they will select one of these generators to add to $S$. Hence, $S = \{g\}$, so $\langle S \rangle = \langle g \rangle = G$. All of $G$ has been generated. Player 1 has won the game. $\square$

# Chapter 3

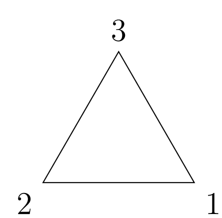
# Dihedral Groups

## 3.1 Introduction

Dihedral groups are often thought of as the symmetries of regular polygons. Imagine assigning a distinct number $1, 2, ..., n$ to each vertex of an $n$-gon. When we think of these symmetries, we are thinking of all of the possible rigid motions of these verticies that can be obtained, begining at a fixed vertex moving clockwise around the $n$-gon. The most basic dihedral group, $D_3$, is composed of reflections and rotations on an equilateral triangle.




Let's consider the triangle above to be our starting point. Now, consider the rotating the triangle 120 degrees:



By using only rotations and reflections, we can come up with five other symmetries of this triangle. We will denote this 120 degree rotational symmetry as $r$.

The elements of any dihedral group are various rotations and flips. A rotation, $r^k$ moves each vertex clockwise to the next vertex $k$ times. Each flip is a different reflection upon an axis of symmetry of the $n$-gon. By definition, each $n$-gon will have $n$ lines of symmetry, and $n$ possible rotations. Thus, a dihedral group $D_n = \{r, r^2, ..., r^{n-1}, r^n = e, f_1, f_2, ..., f_n\}$, where each $r^k$ is a rotation and each $f_l$ is a flip. So for the symmetries of an equilateral triangle, we have $D_3 = \{e, r, r^2, f_1, f_2, f_3\}$.

## 3.2 Three-Player GEN with Dihedral Groups

Fortunately, dihedral groups prove to be a bit more interesting in a game of Three-Player GEN than cyclic groups are. As described in Section 3.1, dihedral groups are composed of two essential components: the rotations and the flips. Before developing a strategy for winning Three-Player GEN with dihedral groups, we must first determine what is needed in order to generate a dihedral group.

For a clearer picture, let's divide the dihedral group, $D_n$, that has been selected into the subgroup of rotations $R_n = \{r, r^2, r^3, \ ..., \ r^n\}$, and the set of rotations carried out on a flipped dihedral. (Note: we are thinking of the flips in this way because of the nature of the game.) To generate $R_n$, we can pick the smallest rotation $r$, a single rotation $r^k$ where $k$ is relatively prime to $n$, or a combination of rotations $r^{m_1}, r^{m_2}, ..., r^{m_x}$ where there exists a pair of $m_1, m_2, ..., m_x$ that are relatively prime. When any of these are obtained, $R_n$ is generated.

To generate the remaining portion of the dihedral group, we simply need to add any one of the flips to the set of elements that generated $R_n$. Because the elements within our set are able to generate $R_n$, they will be able to rotate the flipped dihedral to each required position as well. While flips alone can generate dihedral groups, the nature of the game allows us to ignore these cases. After taking any flip, the next player would just need to select $r$ to win. Thus, the set that generates $D_n$ will contain the following:

1. A single rotation $r^k$ where $k$ is relatively prime to $n$, or a combination of rotations $r^{m_1}, r^{m_2}, ..., r^{m_x}$ where there exists a pair of $m_1, m_2, ..., m_x$ that are relatively prime We will call this set a Relatively Prime Rotation Set, RPRS, for future reference. Since a RPRS can be composed of many different elements, we only care about it when it has been generated. Reminder: It is possible for this set to contain only one element.

2. Any one flip from $D_n$.

**Lemma 3.1.** *The game of Three-Player GEN on a dihedral group $D_n$ will end on the turn that follows a player selecting either any flip or the rotation that causes a RPRS to be in the set of selected elements.*

*Proof.* First, consider the case when a player selects a flip. Then, the next player can simply pick the smallest rotation $r$, and the set is generated. Thus, the game ends.

Now, suppose this is the first time a RPRS is in the set of selected elements. Then $R_n$ is contained in the set of generated elements So, the next player just needs to select any of the flips to generate the entire set. Thus, the game ends.

Therefore, the game of Three-Player GEN on a dihedral group $D_n$ will end on the turn that follows a player selecting either any flip or the rotation that causes a RPRS to be in the set of selected elements. □

## 3.3 Strategy

In the outline of the rules of the game in Section 1.2, we decided that a player would rather make a play to have the player before them win than the player after them. For dihedral groups, this preference is essential to the strategy that players will follow.

Since players do not want the player immediately following them to win, they will do everything in their power to avoid selecting an element that will set up the next player for victory. As outlined in Section 3.2, the two things required to generate a dihedral group $D_n$ are:

1. A set of rotations $r^{m_1}, r^{m_2}, ..., r^{m_k}$ where any two of $m_1, m_2, ..., m_k$ and $n$ are relatively prime—a RPRS.

2. Any one flip from $D_n$.

Due to these requirements and what we defined by Lemma 3.1, we can draw a few conclusions about the strategy that players will follow while playing Three-Player GEN with dihedral groups.

1. Players will avoid selecting an element that completes a RPRS.

2. Players will avoid selecting a flip.

These two strategic rules leave us with only a few viable elements for selection to begin a game, no matter which dihedral group is being used for the game. Obviously, the identity $e$ would be a safe pick for Player 1. In fact, if $D_n$ is a dihedral group in which $n$ is prime, the identity is the only pick that allows Player 1 to avoid completing a RPRS or selecting a flip.

### 3.3.1 Three-Player GEN with $D_n$ where $n$ is Prime

When $n$ is prime, we will find that the game of Three-Player GEN on $D_n$ does not last for many turns. In fact, the game will always be over after exactly three turns under optimal play. Player 1 will always begin the game by selecting the identity $e$, and Player 3 will always be the victor.

**Theorem 3.1.** *When playing Three-Player GEN with a dihedral group $D_n$ in which $n$ is prime, Player 3 has the winning strategy.*

*Proof.* Let $D_n$ be the group used for a game of Three-Player GEN, and let $n$ be prime. By the outline of our strategy, Player 1 will do everything in their power to avoid setting up Player 2 for a win.

By contradiction, we will show that Player 1 must pick the identity $e$ in order to avoid Player 2 winning. Assume that Player 1 does not select the identity $e$ to start the game. Thus, Player 1 will have decided to take either a flip or a non-identity rotation.

By selecting a flip, Player 1 will have fulfilled the requirement of any one flip from $D_n$ to generate the group. By Lemma 3.1, the game will end on the next turn and Player 2 will win.

By selecting a non-identity rotation, Player 1 will generate $R_n$, since all non-trivial elements of $R_n$ are generators when $n$ is prime. By Lemma 3.1, the game will end on the next turn and Player 2 will win.

Therefore, we have a contradiction. The only element available for selection that will not set up the following player for victory in $D_n$ is the identity $e$. Player 1 will select this element, and then Player 2 will be forced to pick an element that, by Lemma 3.1, sets up the victory for Player 3. Thus, Player 3 will always win the game when $n$ is prime. $\square$

### 3.3.2 Three-Player GEN with $D_n$ where $n$ is Composite

When $D_n$ has an $n$ that is composite, our game of Three-Player GEN gets a little more interesting. To identify the winners of the game involving different dihedral groups, we need to take a closer look at our RPRS possibilities. We will not pay much attention to the flips of $D_n$ in this section, as the flips remain trivial to strategy.

Maximal subgroups are the key to playing with a group $D_n$ where $n$ is composite (even though they are key when $n$ is prime as well, since $e$ is maximal in $R_p$). As outlined earlier, a maximal subgroup is one that cannot become any larger without becoming the entire group itself. For this iteration of Three-Player GEN, we are examining the maximal subgroups of the subgroup of rotations in $D_n$.

With a prime $n$ in $D_n$, every rotation $r^k$ that could be selected generated the group of rotations, other than the identity. Hence, the identity was the only "safe" pick to make. With a composite $n$, there are more "safe" picks that can be made.

Consider $D_{12}$. In $D_{12}$, there are 12 elements in the subgroup of rotations: $\{e, r^1, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11}\}$. The following elements are the "safe" ones that Player 1 is able to choose from: $\{e, r^2, r^3, r^4, r^6, r^8, r^9, r^{10}\}$. Aside from the identity $e = r^0$, all of the rotations in this set $r^x$ have an $x$ value that is not relatively prime to the $n$ value of $D_n$. Note that not all of these elements would remain "safe" as the game moves along.

By selecting any of these elements, Player 1 will force Player 2 to continue to build a maximal subgroup of the rotations. As we will see, Player 1 will choose to select an element that is not the identity so that Player 1 decides which maximal subgroup of rotations will be constructed.

Let's examine the maximal subgroups of the rotations of $D_{12}$. They are: $\{e, r^2, r^4, r^6, r^8, r^{10}\}$ and $\{e, r^3, r^6, r^9\}$. By only taking elements from one of these maximal subgroups, the players delay constructing a RPRS. Players 2 and 3 will be forced to continue building one of these maximal subgroups, as they do not want the next player to win, as outlined in Section 1.2.

Since these maximal subgroups are the only elements that can be chosen safely (without constructing a RPRS or picking a flip), we need to identify how their sizes matter. Let $s$ be the size of one of these maximal subgroups. After all of the elements of the subgroup have been picked, $s$ turns will have been taken. Then, either a RPRS will be completed, or a flip will be taken on the next turn. Finally, the opposite will occur on the last turn, and the game will be over. Hence, the game will last $s + 2$ turns. Table 3.1 provides an outline of the number of total turns required for a player to win the game.

| Player 1 | Player 2 | Player 3 |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |
| ... | ... | ... |
| 1(mod 3) | 2(mod 3) | 0(mod 3) |

Table 3.1: Winner of GEN by Number of Total Turns

Let's subtract the two moves that will always be the last two to occur (completing the RPRS and selecting a flip). Now, we have a table of the size of the maximal subgroup of rotations needed to win the game. As was the case with $D_n$ where $n$ is prime, the only maximal subgroup of rotations is $\{e\}$, which has a size of 1. We proved that Player 3 always won in this instance. This follows in Table 3.2 below, which outlines the winner of the game based on the size of the maximal subgroup used.

| Player 1 | Player 2 | Player 3 |
|---|---|---|
| - | - | 1 |
| 2 | 3 | 4 |
| 5 | 6 | 7 |
| ... | ... | ... |
| 2(mod 3) | 0(mod 3) | 1(mod 3) |

Table 3.2: Winner of GEN on $D_n$ by Size of Maximal Subgroup of $R_n$

With $D_{12}$, the size of the two maximal subgroups are 4 and 6. By looking at Table 3.2, we see that this either means Player 2 or Player 3 will win. By Section 1.2, Player 1 would rather have Player 3 win than Player 2. Thus, Player 1 wants to use the maximal subgroup of size 4 for the game. To ensure that this subgroup is used, Player 1 will need to select an element that is unique to this maximal subgroup, like $r^3$. If Player one were to pick $r^6$, Player 2 would be able to determine which maximal subgroup to use, because $r^6$ is an element of both of the maximal subgroups.

For any dihedral group $D_n$, the winner can be predetermined by examining the prime factorization of $n$. There will be as many maximal subgroups of the rotations of $D_n$ as there are unique prime factors $\{p_1, p_2, ..., p_m\}$ of $n$. By dividing $n$ by each unique prime factor, we are able to determine the sizes of each of the maximal subgroups.

After identifying the sizes of these subgroups, Player 1 must select a unique element of the maximal subgroup they would like to construct to begin. To do so, Player 1 can simply take the rotation $r^{p_1}$, where $p_1$ is the prime factor used to identify the size of the maximal subgroup. Clearly, this element cannot exist in any other maximal subgroup, as the rotations in each maximal subgroup are separated by multiples of some prime $p$. That is, a maximal subgroup will be of the form $\{r^p, r^{2p}, r^{3p}, ..., r^{lp}\}$, where $lp = n$ for the group $D_n$. We know that $p_1$ is prime, so it cannot take the form $kq$ for any integer $k$ and prime $q$ except for $k = 1$ and $q = p_1$. Thus, $r^{p_1}$ can only exist in the maximal subgroup of rotations separated by multiples of $p_1$.

The following theorem generalizes Theorem 3.1.

**Theorem 3.2.** *For a game of Three-Player GEN on any dihedral group $D_n$, the winning strategies will be determined by the values of $n/p$ for the prime divisors $p$ of $n$. If there exists a prime $p$ that divides $n$ such that $n/p \equiv 2 \pmod 3$, Player 1 will win. If no such $p$ exists, but there is a $q$ that divides $n$ such that $n/q \equiv 1 \pmod 3$, Player 3 will win. Otherwise, Player 2 will win.*

*Proof.* Let $D_n$ be a dihedral group being used for a game of Three-Player GEN, and let it be the case that there exists a prime $p$ that divides $n$ such that $n/p \equiv 2 \pmod 3$. Thus, we have a maximal subgroup of rotations $M$ with the format: $M = \{r^p, r^{2p}, ..., r^{((n/p)-1)p}, r^{(n/p)p}\}$. Note that $r^{(n/p)p} = e$ and that there are exactly $(n/p)$ elements in the maximal subgroup. Since $n/p \equiv 2 \pmod 3$, $M$ falls under the category in Table 3.2 of maximal subgroups that will lead Player 1 to win the game. Therefore, Player 1 will always win a game on a dihedral group $D_n$ where there exists a prime $p$ such that $n/p \equiv 2 \pmod 3$.

When do we have that $n/p \equiv 2 \pmod 3$? Considering that $n$ is composite, we know that $n/p$ will take this form when either any of the following cases occur, by modular arithmetic:

1. $n \equiv 2 \pmod 3$ and $p \equiv 1 \pmod 3$
2. $n \equiv 1 \pmod 3$ and $p \equiv 2 \pmod 3$
3. $n \equiv 0 \pmod 3$ and $n/3 \equiv 2 \pmod 3$

Now suppose no such prime $p$ exists. If Player 1 is unable to force a win for herself, she will attempt to force a win for Player 3. So, suppose that a prime $q$ exists such that $n/q \equiv 1 \pmod 3$. Then we would have a similar maximal subgroup of rotations with size equivalent to $1 \pmod 3$. Thus, by picking $r^q$, Player 1 will force the desired maximal subgroup to be generated and Player 3 will end up being the winner, by Table 3.2.

Similar to when we were looking for a prime $p$ so that $n/p \equiv 2 \pmod 3$, we know that $n/q \equiv 1 \pmod 3$ under any of the following circumstances:

1. $n \equiv 2 \pmod 3$ and $q \equiv 2 \pmod 3$
2. $n \equiv 1 \pmod 3$ and $q \equiv 1 \pmod 3$
3. $n \equiv 0 \pmod 3$ and $n/3 \equiv 1 \pmod 3$

Finally, consider the case in which for any prime $p$ that divides $n$, we have that it is neither the case that $n/p \equiv 1 \pmod 3$ nor $n/p \equiv 2 \pmod 3$ for all prime divisors $p$ of $n$. That is, all prime divisors $p_i$ of $n$ give us $\dfrac{n}{p_i} = 3k_i$ for integers $k_i$. Thus, Player 1 has no choice but to pick an element that will generated a maximal subgroup of rotations of size equivalent to $0 \pmod 3$. Then, by Table 3.2, we have that Player 2 will be the victor.

Therefore, the winner of a game of Three-Player GEN on a dihedral group $D_n$ is determined by the values of $n/p$ for the prime divisors $p$ of $n$. If there exists a prime $p$ such that $n/p \equiv 2 \pmod 3$, Player 1 will win. If no such $p$ exists, but there is a $p$ such that $n/p \equiv 1 \pmod 3$, Player 3 will win. Otherwise, Player 2 will win. $\square$

# Chapter 4

# Nilpotent Groups

## 4.1 Introduction

### Nilpotent Groups

A nilpotent group $N$ is a group that is isomorphic to a direct product of $p$-groups, where a $p$-group is a group that has $p^z$ elements, where $z$ is a positive integer. So, we have that $N \cong P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$, where each $P_{a_i}$ is a $p$-group and $a_i$ are primes. In this game, we will go a step further, denoting each $P_{a_i}$ as a Sylow $p$-subgroup, which is the largest proper subgroup of order $p$ of the group $N$. Thus, $p$-groups that share the same prime $p$ will be put together into one Sylow $p$-subgroup factor $P_{a_i}$ when we look at the direct product of $N$. For simplicity, we will always refer to this direct product of Sylow $p$-subgroups when talking about a nilpotent group $N$.

### Abelian Groups

An abelian group is a special type of nilpotent group that is defined as a group $A$ with an operation $*$ such that for every $x, y \in A$, it is the case that $x * y = y * x$. From Theorem 7.10 of Isaacs[5], the Fundamental Theorem of Abelian Groups states that each finite abelian group is isomorphic to the direct product of cyclic groups that have orders of prime powers. That is, $A \cong \mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times ... \times \mathbb{Z}_{{p_n}^{n_m}}$, where the $p_i$ are prime and not necessarily distinct, and the $n_j$ are integers. This theorem will allow us to better understand how these types of group work in a game of Three-Player GEN.

For all nilpotent groups $N$, we will use the notation of $d(N) = n$, where $n$ is the minimum number of elements required to generate $N$. Thus, if $d(N) = 2$, the group can be generated by 2 elements, but no fewer than 2 elements.

## 4.2 Three-Player GEN with Nilpotent Groups where $d(N) = 2$

Consider the nilpotent group $N = P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$. By definition, each of the factors of $N$ is a Sylow $p$-subgroup. That is, each of the factors $P_{a_i}$ has a size of ${p_i}^{n_i}$, where $p_i$ is a prime and $n_i$ is a positive integer. So, $P_{a_1}$ has a size such that some prime $p_{a_1}$ divides it, as does $P_{a_2}$ with $p_{a_2}$, and so on. As we noticed before in our examination of dihedral groups, "safe elements" will be the key in determining who will win in a game of GEN on nilpotent groups.

For any group that we are using for a game of GEN, the main goal is simple: win. The second goal is: do not allow the next player to win. To do this, players need to avoid selecting elements of the group that open the door for the last needed generator to be chosen. With nilpotent groups, there can be many generators. However, the number of generators does not change the strategy of the players: they need to make sure that they do not select the second to last generator. So, the process of playing a game of GEN with any nilpotent group will proceed as follows:

1. Players take turns selecting elements until the "safe elements" and "unsafe elements" are determined.

2. Players avoid the "unsafe elements" that would be the second to last generator needed in the common pool, $S$.

3. Players select all remaining "safe elements" in the group.

4. Once all of the "safe elements" are gone, a player is forced to pick an "unsafe element."

5. The next player wins the game by selecting the final generator.

Recall from Section 1.1 that a safe set is defined as a set $H$ such that $\langle H, x\rangle < G$ for all $x \in G$.

**Proposition 4.1.** *For a group $G$, a set of safe elements $K$ and any element $x \in G$, it is the case that $\langle\langle K\rangle, x\rangle = \langle K, x\rangle$.*

*Proof.* Clearly, we have that $\langle K, x\rangle \leq \langle\langle K\rangle, x\rangle$, since $K \subseteq \langle K\rangle \subseteq \langle\langle K\rangle, x\rangle$ and $x \in \langle\langle K\rangle, x\rangle$. Now, $x \in \langle K, x\rangle$ and $K \subseteq \langle K, x\rangle$, so $\langle K\rangle \subseteq \langle K, x\rangle$. So $\langle\langle K\rangle, x\rangle \subseteq \langle K, x\rangle$. Thus, $\langle\langle K\rangle, x\rangle = \langle K, x\rangle$. □

**Proposition 4.2.** *A set of safe elements $K$ of a group $G$ is a subgroup of $G$.*

*Proof.* Let $K$ be the set of safe elements of a group $G$. We will prove that $K = \langle K\rangle$. Since $K$ is safe, we know that for all $x \in G$, it is the case that $\langle K, x\rangle < G$. Then $\langle\langle K\rangle, x\rangle = \langle K, x\rangle < G$ for all $x \in G$, so $\langle K\rangle$ is safe. So $\langle K\rangle = K$. Therefore, since $\langle K\rangle$ is a subgroup, $K$ is a subgroup. □

**Theorem 4.3.** *A set of safe elements in a game of Three-Player GEN on a nilpotent group $N = P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$ will be equivalent to the subgroup $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times K$, where $\{Q_{b_1}, ..., Q_{b_m}\} = \{P_{a_1}, ..., P_{a_m}\}$ and $K$ is a proper subgroup of $Q_{b_m}$.*

*Proof.* Let $N = P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$ be a nilpotent group used for a game of Three-Player GEN. We can conclude that the last two generators in a game of GEN on a nilpotent group will come from the same $p$-group. If they were to come from separate $p$-groups, a player would be able to select them both at the same time since $N$ is a direct product with coprime factor orders. Thus, the safe elements that the players will have to chose from will be the direct product of all but one of the $p$-groups, as well as a separate set $K$ that comes from the $p$-group that contains the final two generators.

So, the safe elements $H$ for $N$ will be $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times K$, where each $Q_{b_i}$ is a $p$-group factor that does not contain the final two generators, and $K$ is a subgroup of safe elements coming from $Q_{b_m}$, the factor with the final two generators. We know that $K$ is a subgroup by Proposition 4.2. Thus, since $K$ is a subgroup of $Q_{b_m}$, we have that $H$ is a subgroup of $G$. So $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times K$ is the subgroup of safe elements elements for the group $N$. □

The following corollary, Corollary 4.4, is a result of the proof from above.

**Corollary 4.4.** *The final two elements chosen in a game of Three-Player GEN on a nilpotent group $N$ will come from the same factor $Q_{b_m}$.*

For the majority of the rest of the paper, we will focus on nilpotent groups with $d(N) = 2$. With these particular groups, we are able to draw clearer conclusions about winning strategies.

**Corollary 4.5.** *If $d(N) = 2$, then there is at least one $Q_{b_i}$ with $d(Q_{b_i}) = 2$. For one of these, $Q_{b_m}$, we will have $K = \Phi(Q_{b_m})$. The safe set is $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times \Phi(Q_{b_m})$.*

*Proof.* Let $g \in Q_{b_m}$ and $v \in \Phi(Q_{b_m})$. If $\langle v, g\rangle = Q_{b_m}$, then $\langle v, g\rangle \subseteq \langle \Phi(Q_{b_m}), g\rangle$, so $Q_{b_m} = \langle \Phi(Q_{b_m}), g\rangle$ which implies that $Q_{b_m} = \langle g\rangle$ (Isaacs[5,27]). However, this is a contradiction, because $d(Q_{b_m}) = 2$. Thus, $\langle v, g\rangle < (Q_{b_m})$.

Now, let $v \notin \Phi(Q_{b_m})$. For simplicity, let $P = Q_{b_m}$ and $F = \Phi(Q_{b_m})$. Then $Fv \neq F$ in $P/F \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where the isomorphism follows from the Burnside Basis Theorem (Rose[7,274]). Since $Fv \neq F$, there exists an $Fy \in P/F$ such that $\langle Fv, Fy\rangle = P/F$. Let $H = \langle v, y\rangle \subseteq P$. If $H < P$, then $H \leq M$ for some maximal subgroup, $M$. Then $P/F = \langle Fv, Fy\rangle \subseteq MF/F = M/F < P/F$, where $MF = M$ since $F$ is the intersection of all maximal subgroups. We obviously cannot have that $P/F < P/F$, so this is a contradiction. Thus, it must be the case that $\langle v, y\rangle$ is not contained in any maximal subgroup, and is therefore equal to $P$. Thus, $v$ is not safe.

Thus, $v \in \Phi(Q_{b_m})$ is safe and $v \notin \Phi(Q_{b_m})$ is unsafe. Therefore, the safe elements coming from $Q_{b_m}$ are equal to $\Phi(Q_{b_m})$. So, for a game on a nilpotent group $N$ with $d(N) = 2$, the safe elements that players can choose from will be $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times \Phi(Q_{b_m})$. ☐

### 4.2.1 Nilpotent Groups With Sizes Divisible by Three

Before we dive into more specific cases of nilpotent groups, we need to make an observation that will simplify winning the game of GEN on nilpotent groups for one of our players. As outlined in the introduction to this section, all nilpotent groups are isomorphic to a direct product of $p$-groups. Due to this fact, we are able to draw a separate conclusion about nilpotent groups that have sizes that are divisible by three.

As was the case with dihedral groups, if the number of safe elements $H \equiv$

0(mod 3), Player 2 will be the victor. Player 3 will be the last player to select a safe element, Player 1 will pick an unsafe element, and Player 2 will pick the final generator to win. Due to the makeup of nilpotent groups, it is easy to identify when $H \equiv 0(\text{mod } 3)$.

Assume that there exists a factor $P_{a_i}$ of $N = P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$ such that $P_{a_i}$ is a Sylow 3-subgroup. Then, $|P_{a_i}| = 3^n$ for some $n$. So, if $P_{a_i}$ does not contain the final two generators of $N$, $|P_{a_i}|$ will be a factor multiplied to obtain the number of safe elements $|H|$. Since 3 divides $P_{a_i}$, it is also the case that $3||H|$. So, $|H| \equiv 0(\text{mod } 3)$ and Player 2 will win. In the case that the 3-group $P_{a_i}$ is the $p$-group that contains the final two generators, there is only one case in which Player 2 will be unable to win the game.

**Theorem 4.6.** *In a game of Three-Player GEN on a nilpotent group $N$ with a Sylow 3-subgroup $P_3$, if it is the case that $3 \mid |N|$:*

- *Player 1 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 2(\text{mod } 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*
- *Player 3 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 1(\text{mod } 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*
- *Player 2 will win otherwise.*

*Proof.* Let $N = P_{a_1} \times P_{a_2} \times P_{a_3} \times ... \times P_{a_m}$. Without loss of generality, let $P_{a_1} = P_3$. Think of the format of each element chosen by players as $(x_1, x_2, x_3, ..., x_m)$. If Player 1 picks $\bar{x} = (x_1, x_2, x_3, ..., x_m)$ such that $3|o(x_1)$, then either Player 2 picks $\bar{y}$ such that $\langle \bar{x}, \bar{y} \rangle = N$ and wins, or the safe set $H$ is divisible by 3 and Player 2 will win. So Player 1 will not select an element like $\bar{x}$. Note that if $d(N) = 1$, then $N$ is Cyclic, and Player 1 will win.

We can assume that Player 1 will select an element $\bar{x} = (1, x_2, x_3, ..., x_m)$, where 1 is the identity of $P_3$. Then Player 2 will select $\bar{z} = (z, 1, 1, ..., 1)$, where $1 \neq z \in \Phi(P_3)$ if $\Phi(P_3) \neq 1$ and $1 \neq z$ otherwise. Player 3 will pick an element $\bar{y} = (y_1, y_2, y_3, ..., y_m)$.

If $d(P_3) \geq 3$, we have that $\langle 1, z, y \rangle = \langle z, y \rangle < P_3$, so $N$ is not generated. Thus, Player 2 is able to select $\bar{z}$ and make it the case that $3||H|$. Thus, Player 2 will win.

If $d(P_3) = 2$ and $P_3 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$, then we have $\langle 1, z, y \rangle = P_3$ implies $\langle y \rangle = P_3$ Isaacs[5]. Clearly, this is a contradiction, because $d(P_3) = 2$. So Player 2 is able to select $\bar{z}$ and force $|H|$ to be divisible by 3. So Player 2 will win.

If $d(N) \geq 3$, there exists an $i$ such that $d(P_{a_i}) \geq 3$. Then, we have that $\langle x_i, 1, y_i \rangle = \langle x_i, y_i \rangle < P_{a_i}$, so $N$ has not been generated. Thus, Player 2 is able to select $\bar{z}$ as his first element. Then Player 2 will win.

Now, if $d(N) = 2$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, Player 1 will once again pick $\bar{x}$, but Player 2 is unable to select $\bar{z}$ such that $\bar{z} \neq e$, the identity of $N$. By Corollary 4.5, $K = \Phi(P_3) = \{1\}$. Since $|P_3| = 9$, then $|H| = \dfrac{|N|}{9}$. Thus, if $\dfrac{|N|}{9} \equiv 1 \pmod 3$, Player 3 will win. Similarly, if $\dfrac{|N|}{9} \equiv 2 \pmod 3$, Player 1 will win.

Finally, consider when $d(P_3) = 1$ and $d(N) \geq 2$. Clearly, for any $\bar{x}$ that Player 1 picks, there must be an $i$ such that $P_{a_i}$ is not able to be generated by Player 2. That is, Player 1 must pick $x_i$ such that $\langle x_i, z_i \rangle < P_{a_i}$. So, Player 2 is able to select an element with a generator of $P_3$ in his first selection and the identity for the $i$th term. Thus, $N$ cannot be generated when Player 3 selects $\bar{y}$. So, $|H|$ is divisible by 3. Thus, Player 2 will win.

Therefore, we have that the winning strategies of a game of Three-Player GEN on a nilpotent group $N$ such that $3 \mid |N|$ are as follows:

- Player 1 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 2 (\mathrm{mod}\ 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
- Player 3 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 1 (\mathrm{mod}\ 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
- Player 2 will win otherwise.

$\square$

### 4.2.2 Abelian Groups where $d(G) = 2$ and $3 \nmid |G|$

As we progress forward into our examination of groups in the game of Three-Player GEN, taking a moment to look at abelian groups will help us better understand nilpotent groups.

As defined by Judson, "Suppose that $G$ is a[n] [abelian] group and let $\{G_i : i \in I\}$ be a set of elements in $G$, where $i$ is some index set $I$ (not necessarily finite). The smallest subgroup of $G$ containing all of the $g_i$'s is the subgroup of $G$ generated by the $g_i$'s. If this subgroup of $G$ is in fact all of $G$, then $G$ is generated by the set $\{g_i : i \in I\}$. In this case the $g_i$'s are said to be the generators of $G$. If there is a finite set $\{g_i : i \in I\}$ that generates $G$, then $G$ is finitely generated"(Judson 197).

In this rendition of the game, players will need to successfully select all of the elements contained in a finite set $\{g_i : i \in I\}$ that generates $G$. Players will need to identify which of these finite sets are being constructed and make sure that they do not pick the second to last element needed to complete the set.

Due to the amount of cyclic groups that can form the direct product that is isomorphic to a finite abelian group, we will break them down into separate cases to explain their strategies. We will begin with direct products of the form $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$, where $p$ is a prime number and $i$ and $j$ are integers. Later on, we will explain the strategy for all abelian groups in general. Note: We are skipping the case of $\mathbb{Z}_{p_0} \times \mathbb{Z}_{p_1}$ where $p_0$ and $p_1$ are distinct primes, because groups of that form are cyclic.

### 4.2.3 Abelian Groups: $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ and $p \neq 3$

For a finite abelian group isomorphic to $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ where $p$ is a prime number and $i$ and $j$ are integers, our game depends completely on the integers $i$ and $j$. Since $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ involves a direct product of two cyclic groups, we know that both of them need to be independently generated in order for the entire group to be generated. Consider $\mathbb{Z}_{p^i}$ as the $x$-coordinate and $\mathbb{Z}_{p^j}$ as the $y$-coordinate in a $(x, y)$ ordered pair. Players are going to be selecting ordered pairs to add to the common pool $S$ to try and generate the group.

Since this type of abelian group is not cyclic, players will need to use a combination of elements to generate the group. Players will need to generate both the $x$-coordinate and $y$-coordinate, but they must also be able to form all $(x, y)$ combinations.

For example, if Player 1 were to select $(0,1)$ to begin the game, he would not be able to win. The generated subgroup would be $S_1 = \langle (0,1) \rangle = \{(0,1),(0,2),...,(0,p^{j-1}),(0,p^j)\} \neq \mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$. Player 1 would have successfully generated $\mathbb{Z}_{p^j}$, but not $\mathbb{Z}_{p^i}$. However, it is not sufficient to pick an element like $(1,1)$ to win the game. While both of the factors will be generated, they are not independently generated. Clearly, an element such as $(0,1)$ would not be in $S_1 = \langle (1,1) \rangle$.

As we discussed with dihedral groups, players are going to be looking to select "safe" elements to add to the common pool, $S$. If Player 1 begins the game by picking any element that generates the $x$-coordinate, Player 2 simply needs to select $(0,1)$ to add to $S$ to win the game. Similarly, if Player 1 begins the game by picking any element that generates the $y$-coordinate, Player 2 will select $(1,0)$ to add to $S$ and win the game.

So, Player 1 is restricted to picking an element that does not generate either of the coordinates. Let's examine the safe elements of an abelian group that is isomorphic to $\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}$:

| | | | |
|---|---|---|---|
| (0,0) | (0,2) | (0,4) | (0,6) |
| (2,0) | (2,2) | (2,4) | (2,6) |
| (4,0) | (4,2) | (4,4) | (4,6) |
| (6,0) | (6,2) | (6,4) | (6,6) |
| (8,0) | (8,2) | (8,4) | (8,6) |
| (10,0) | (10,2) | (10,4) | (10,6) |
| (12,0) | (12,2) | (12,4) | (12,6) |
| (14,0) | (14,2) | (14,4) | (14,6) |

Table 4.1: Safe Elements in GEN on $\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}$

As we can see in Table 4.1, there are 32 safe elements. By looking a little closer, we can see that $2^{4-1} \times 2^{3-1} = 2^3 \times 2^2 = 8 \times 4 = 32$. This notation of $2^{4-1} \times 2^{3-1}$ is important: each of the factors represent the sizes of the Frattini subgroups of $\mathbb{Z}_{2^4}$ and $\mathbb{Z}_{2^3}$, respectively. This is no surprise, since by Corollary 4.5, we have that $K = \Phi(\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j})$. It follows from Dlab[4] that $\Phi(\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}) = \Phi(\mathbb{Z}_{p^i}) \times \Phi(\mathbb{Z}_{p^j})$.

By the Fundamental Theorem of Abelian Groups, we know that the abelian group is isomorphic to $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$. So the elements in Table 4.2 will be safe.

| | | | |
|---|---|---|---|
| (0,0) | (0,$p$) | ... | (0,$(p^{j-1}-1)p$) |
| ($p$,0) | ($p$,$p$) | ... | ($p$,$(p^{j-1}-1)p$) |
| ($2p$,0) | ($2p$,$p$) | ... | ($2p$,$(p^{j-1}-1)p$) |
| ($3p$,0) | ($3p$,$p$) | ... | ($3p$,$(p^{j-1}-1)p$) |
| ... | ... | ... | ... |
| ($(p^{i-1}-1)p$,0) | ($(p^{i-1}-1)p$,$p$) | ... | ($(p^{i-1}-1)p$,$(p^{j-1}-1)p$) |

Table 4.2: Safe Elements in GEN on $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$

We have a table of "safe" elements that is $p^{j-1}$ columns by $p^{i-1}$ rows. So, there are exactly $p^{j-1} \times p^{i-1}$ elements that can be added to $S$ without the next player being able to select an element that will win the game. Thus, the sizes of the Frattini subgroups are multiplied together to determine the number of safe elements.

Since players want to avoid having the following player win the game, all elements of $\Phi(\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j})$ will be chosen before a player must choose a separate element. Similar to the dihedral groups, two elements (each of which are relatively prime to at least one of the coordinates) will be added to the common pool $S$ after all of the safe elements are chosen, and the game will end.

Recall that in Section 4.3, we discussed the results of having a group such that $3 \mid G$. The theorem below covers all groups $G = \mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ where $p \neq 3$.

**Theorem 4.7.** *The winner of the game of GEN on $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ where $i, j \neq 0$ will be determined by the values of the prime $p \neq 3$ and $i + j$. If $p \equiv 1 (mod\ 3)$, Player 3 will always win. If $p \equiv 2\ (mod\ 3)$ and $i+j$ is odd, then Player 3 will win. If $p \equiv 2\ (mod\ 3)$ and $i + j$ is even, then Player 1 will win.*

*Proof.* Let $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ be the abelian group used for a game of GEN with three players. First, consider the case where $p \equiv 1 \pmod 3$. Then we know that the size of the group will be equivalent to 1 (mod 3), as primes of this sort will remain equivalent to 1 (mod 3) no matter what power they are raised to. Similarly, since the size of Frattini subgroup will be equivalent to $\mathbb{Z}_{p^{i-1}} \times \mathbb{Z}_{p^{j-1}}$, it will also have a size equivalent to 1 (mod 3), as the size is just divided by $p^2 \equiv 1 \pmod 3$. Thus, the number of safe elements, which is equal to the number of elements in the Frattini subgroup, will be equivalent to 1 (mod 3). Thus, Player 3 will be the winner.

Now, consider when $p \equiv 2 \pmod 3$. Unlike before, $p^n$ will oscillate between being equivalent to 1 (mod 3) and 2 (mod 3), depending on the parity of $n$. By modular arithmetic, for a prime $p \equiv 2 \pmod 3$, $p^2 \equiv 2 \times 2 \equiv 4 \equiv 1 \pmod 3$. Similarly, as another $p$ is multiplied, $p^3 \equiv 2 \times 2 \times 2 \equiv 8 \equiv 2 \pmod 3$. This continues. So if $n$ is odd, $p^n \equiv 2 \pmod 3$. Likewise, if $n$ is even, $p^n \equiv 1 \pmod 3$. Thus, since $p^i$ and $p^j$ are being multiplied by each other, the size of the group will be $p^{i+j}$. By letting $n = i + j$, we see that if $i + j$ is odd, the size of the group will be equivalent to 2 (mod 3), and if $i + j$ is even, the size of the group will be equivalent to 1 (mod 3).

Now, notice that the size of the Frattini subgroup will be equal to $p^{i-1} \times p^{j-1} = p^{i+j-2}$. Also see that subtracting 2 from $i+j$ does not change whether it is even or odd. So, we know that the size of the Frattini subgroup will be equivalent to the same number (mod 3) as the size of the entire group. So, if $i + j$ is odd, then $i + j - 2$ is also odd, and the size of the Frattini subgroup will be equivalent to 1 (mod 3). Thus, Player 3 will win. If $i+j$ is even, then $i + j - 2$ is also even, and the size of the Frattini subgroup will be equivalent to 2 (mod 3). Thus, Player 1 will win.

Therefore, the winner of GEN on an abelian group of the form $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ where $i, j \neq 0$ will be determined by the values of $p$, $j$, and $i$. If $p \equiv 1$(mod 3), Player 3 will always win. If $p \equiv 2 \pmod 3$ and $i+j$ is odd, then Player 3 will win. If $p \equiv 2 \pmod 3$ and $i + j$ is even, then Player 1 will win. $\square$

### 4.2.4 Abelian Groups: $(\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j}$ where $p \nmid n$

Now, let's examine a family of abelian groups that is isomorphic to $(\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j}$ where $p \nmid n$ and $p \neq 3$. These groups are very similar to the ones that were just covered, but they have an added factor to the isomorphism: $\mathbb{Z}_n$, where $p \nmid n$. This factor plays a major role in determining which player will win.

It is important to make the distinction that $p \nmid n$. Due to this, the $\mathbb{Z}_n$ factor and either the $\mathbb{Z}_{p^i}$ factor or the $\mathbb{Z}_{p^j}$ factor can both be generated at the same time. (By selecting (1,1,0), they will generate all possible combinations of the first two coordinates.) Thus, if any player generates either the $\mathbb{Z}_{p^i}$ factor or the $\mathbb{Z}_{p^j}$ factor (or both), the next player will win on the next move. So, it is easy to see that the size of $\mathbb{Z}_n$ will have a big influence on the game. As we identified earlier in Table 4.1, the winner of the $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ game was dependent on the size of the Frattini subgroup of the group, $\Phi(\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j})$. Once again, we need to look at the Frattini subgroup to help us identify the winner. Similar to before, we have both the $\mathbb{Z}_{p^i}$ and $\mathbb{Z}_{p^j}$ factors. If either is generated (as was the case previously), the game will end on the next turn. So, we will use their respective Frattini subgroup sizes, $p^{i-1}$ and $p^{j-1}$, in determining the Frattini subgroup, $\Phi((\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j})$. We know that there will be $p^{i-1} \times x \times p^{j-1}$ safe elements for our group. We need to identify what value $x$ will have.

We know that $\mathbb{Z}_n$ can be generated by a player without leading to a win for the next player, given that both of the other factors, $\mathbb{Z}_{p^i}$ and $\mathbb{Z}_{p^j}$, have not been generated. For example, assume that Player 1 selected the element $(0, 1, 0)$ to add to the common pool, $S$. It is clear that the $\mathbb{Z}_n$ factor would be generated, but Player 2 would not have the ability to win the game on their turn. The format of all safe elements is:

$$(\text{Element of } \Phi(\mathbb{Z}_{p^i}), \text{Element of } \mathbb{Z}_n, \text{Element of } \Phi(\mathbb{Z}_{p^j}))$$

So, we can see that our $x$ value for determining the number of safe elements will actually be the size of $\mathbb{Z}_n$, $n$. Thus, we will have $p^{i-1} \times n \times p^{j-1}$ safe elements for the players to choose from. Once all of those elements are chosen, a player will be forced to select an unsafe element and generate $\mathbb{Z}_{p^i}$ or $\mathbb{Z}_{p^j}$. The next player will win. Below is the table of winners based on the value of $p^{i-1} \times n \times p^{j-1}$:

| Player 1 | Player 2 | Player 3 |
|---|---|---|
| - | - | 1 |
| 2 | 3 | 4 |
| 5 | 6 | 7 |
| 8 | 9 | 10 |
| 11 | 12 | 13 |
| ... | ... | ... |
| 2(mod 3) | 0(mod 3) | 1(mod 3) |

Table 4.3: Winner of GEN on $(\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j}$ by value of $p^{i-1} \times n \times p^{j-1}$

**Theorem 4.8.** *The winner of the game of Three-Player GEN on* $(\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j}$ *will be determined by the values of* $n$, $p$, *and* $i+j$. *Player 1 will win when any of the following occur:*

1. $p \equiv 1(mod\ 3)$, $n \equiv 2(mod\ 3)$
2. $p \equiv 2(mod\ 3)$, $n \equiv 1(mod\ 3)$ *and* $i+j$ *is odd*
3. $p \equiv 2(mod\ 3)$, $n \equiv 2(mod\ 3)$ *and* $i+j$ *is even*

*Similarly, Player 3 will win when any of these conditions are met:*

1. $p \equiv 1(mod\ 3)$, $n \equiv 1(mod\ 3)$
2. $p \equiv 2(mod\ 3)$, $n \equiv 1(mod\ 3)$ *and* $i+j$ *is even*
3. $p \equiv 2(mod\ 3)$, $n \equiv 2(mod\ 3)$ *and* $i+j$ *is odd*

*Proof.* For the entire proof, consider the results from Theorem 4.7 to understand how $p^i$ and $p^j$ will behave in this group. The only difference is that we have added another factor, $\mathbb{Z}_n$.

First, consider the claimed cases when Player 1 will win. If we have that $p \equiv 1(\text{mod } 3)$ and $n \equiv 2(\text{mod } 3)$, we know that $p^{l+k} \equiv 1(\text{mod } 3)$ for any $l+k$. So, we have that the number of safe elements is $1 \times 2 \equiv 2(\text{mod } 3) \equiv |H|$. So Player 1 will win.

Consider when we have $p \equiv 2(\text{mod } 3)$, $n \equiv 1(\text{mod } 3)$ and $i+j$ is odd. We know from Theorem 4.7 that $p^{i+j-2} \equiv 2(\text{mod } 3)$, so we have $2 \times 1 \equiv 2(\text{mod } 3) \equiv |H|$. So Player 1 will win.

Now consider when $p \equiv 2(\text{mod } 3)$, $n \equiv 2(\text{mod } 3)$ and $i+j$ is even. By Theorem 4.7, $p^{i+j-2} \equiv 1(\text{mod } 3)$. Thus, since $n \equiv 2(\text{mod } 3)$, $2 \times 1 \equiv 2(\text{mod } 3) \equiv |H|$. Player 1 wins this case.

We have similar proofs for Player 3 winning. First, when we have $p \equiv 1(\text{mod } 3)$, $n \equiv 1(\text{mod } 3)$, we know that $p^{l+k} \equiv 1(\text{mod } 3)$ for any $l+k$. So, we have that the number of safe elements is $1 \times 1 \equiv 2(\text{mod } 3) \equiv |H|$. Player 3 will win.

Next, examine the case when $p \equiv 2(\text{mod } 3)$, $n \equiv 1(\text{mod } 3)$ and $i+j$ is even. We know that $p^{i+j-2} \equiv 1(\text{mod } 3)$, so we have $2 \times 1 \equiv 1(\text{mod } 3) \equiv |H|$.

Player 3 wins.

Finally, consider when $p \equiv 2 (\text{mod } 3)$, $n \equiv 2 (\text{mod } 3)$ and $i + j$ is odd. Then we have $p^{i+j-2} \equiv 2 (\text{mod } 3)$. So $2 \times 2 \equiv 1 (\text{mod } 3) \equiv |H|$. Player 3 wins this scenario.

Therefore, we can see that the winner of the game of Three-Player GEN on $(\mathbb{Z}_{p^i} \times \mathbb{Z}_n) \times \mathbb{Z}_{p^j}$ is determined by the values of $n$, $p$, and $i + j$. Player 1 and Player 3 win in their respective cases that are described in the theorem. $\square$

### 4.2.5 Nilpotent Groups where $d(N) = 2$ and $3 \nmid |N|$

As was outlined in earlier in the chapter, if there exists a Sylow 3-subgroup $P_i$ in a nilpotent group, then Player 2 is most likely going to be the victor of the game. (As a reminder: Player 2 will not have this guaranteed victory if $P_i \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $d(N) = 2$.) So, this begs the question: who wins if there is no nontrivial 3-group in the composition of our nilpotent group, $N$? Note: The results of this section generalize the previous, more specific cases of nilpotent groups that we have covered.

Based on our knowledge of how the game works, it is fairly easy to deduce that if $N$ does not have a Sylow 3-subgroup that will lead to a Player 2 victory, Player 2 has no way to win the game. Since $N$ is composed of Sylow $p$-subgroups, there cannot be a product of all but one of the $p$-group sizes and a subgroup $K$ from the final $p$-group $P_{a_k}$ that has order divisible by three. As discussed earlier, none of the sizes of the $p$-groups will be divisible by three, and $|K|$ will be an integer that is equivalent to the size of the Frattini subgroup of $\Phi(P_{a_k})$. Clearly, $|\Phi(P_{a_k})| \not\equiv 0 (\text{mod } 3)$ if $|P_{a_k}| \not\equiv 0 (\text{mod } 3)$.

So, consider a nilpotent group $N$ such that $3 \nmid |N|$. Thus, Player 2 is going to be unable to win the game of Three-Player GEN with this group. Since $d(N) = 2$, every $p$-group that makes up $N$ is at most generated by two elements. Therefore, Player 1 will be able to determine which elements are safe and unsafe before Player 3 takes his or her first turn. We have already determined the possible outcomes of the game when $|N| \equiv 0 (\text{mod } 3)$. To finalize our examination of nilpotent groups where $d(N) = 2$, we must consider when $|N| \not\equiv 0 (\text{mod } 3)$.

We have already used the Burnside Basis Theorem (Rose[7,274]) to assist us in determining the format of the number of safe elements in a game of the Three-Player GEN on a nilpotent group $N$ with $d(N) = 2$. By examining this theorem further, we are able to see that when $3 \nmid |N|$ and $d(Q_{b_m}) = 2$ for a Sylow $p$-subgroup:

$$\begin{aligned} d(Q_{b_m}) = 2 &\implies Q_{b_m}/\Phi(Q_{b_m}) \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ &\implies \frac{|Q_{b_m}|}{|\Phi(Q_{b_m})|} = p^2 \\ &\implies |Q_{b_m}| = p^2|\Phi(Q_{b_m})| \\ &\implies |Q_{b_m}| \equiv |\Phi(Q_{b_m})| \pmod 3 \end{aligned}$$

So, since the set of safe elements $H = Q_{b_1} \times Q_{b_2} \times Q_{b_3} \times ... \times Q_{b_{m-1}} \times \Phi(Q_{b_m})$ by Corollary 4.5 and $|Q_{b_m}| \equiv |\Phi(Q_{b_m})| \pmod 3$, we can deduce that $|H| \equiv |N| \pmod 3$. The following theorem summarizes the aforementioned and allows us to conclude the nilpotent case when $d(N) = 2$.

**Theorem 4.9.** *In a game of Three-Player GEN on a nilpotent group $N$ such that $d(N) = 2$ and $3 \nmid |N|$, Player 1 will win when $|N| \equiv 2$(mod 3) and Player 3 will win when $|N| \equiv 1$(mod 3).*

*Proof.* Let $N$ be a nilpotent group used for a game of Three-Player GEN such that $d(N) = 2$ and $3 \nmid |N|$. As was shown above, for any Sylow $p$-subgroup with $d(Q_{b_m}) = 2$, $|Q_{b_m}| \equiv |\Phi(Q_{b_m})| \pmod 3$.

So, consider the set of safe elements $H$. We have $H = Q_{a_1} \times Q_{a_2} \times Q_{a_3} \times ... \times Q_{a_{m-1}} \times \Phi(Q_{b_m})$ by Corollary 4.5. As shown above with the use of the Burnside Basis Theorem, we know that for any $Q_{b_m}$ chosen by Player 1, $|Q_{b_m}| \equiv |\Phi(Q_{b_m})| \pmod 3$ since $d(Q_{b_m}) = 2$. Thus, we have that we have that:

$$|H| = \frac{|N|}{|Q_{b_m}|}|\Phi(Q_{b_m})| \equiv |N| \pmod 3$$

Therefore, it is clear that when $|N| \equiv 2 \pmod 3$, we will have that $|H| \equiv 2 \pmod 3$ and Player 1 will win. Similarly, when $|N| \equiv 1 \pmod 3$, it will be the case that $|H| \equiv 1 \pmod 3$, so Player 3 will win. □

### 4.2.6 Nilpotent Groups where $d(N) = 2$

Now that we have separately identified the winning strategies for a nilpotent group $N$ when $3 \mid |N|$ and $3 \nmid |N|$, we can combine the theorems to completely span the case of when $d(N) = 2$.

**Theorem 4.10.** *The winner of the game of GEN on a nilpotent group $N$ such that $d(N) = 2$ can be determined by the value of $|N|$, with the exception of when $N$ has a Sylow 3-group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Specifically:*

- *If $|N| \equiv 1 (mod\ 3)$, Player 3 wins.*
- *If $|N| \equiv 2 (mod\ 3)$, Player 1 wins.*
- *If $|N| \equiv 0 (mod\ 3)$ and $P_3 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$, Player 2 wins.*
- *If $|N| \equiv 0 (mod\ 3)$, $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\frac{|N|}{9} \equiv 1 (mod\ 3)$ Player 3 wins.*
- *If $|N| \equiv 0 (mod\ 3)$, $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\frac{|N|}{9} \equiv 2 (mod\ 3)$ Player 1 wins.*

# Chapter 5

# Conclusion

We have seen that the traits and characteristics of the families of groups we have examined have led to many different strategies in the game of Three-Player GEN. While some families of groups had more obvious winning strategies than others (cyclic groups), we found more complex strategies that relied heavily on the composition of each group.

Being able to identify a strategy that works in most cases for nilpotent groups where $d(N) = 2$ was a huge breakthrough, and it turned out to be even better when we were able to apply it to abelian groups as it covered all cases. As a summary, we have come up with the following theorems and corollaries for the game of Three-Player GEN on groups:

### Cyclic Groups

**Theorem 5.1.** *Player 1 has the winning strategy of Three-Player GEN with a cyclic group.*

### Dihedral Groups

**Theorem 5.2.** *For a game of Three-Player GEN on any dihedral group $D_n$, the winning strategies will be determined by the values of $n/p$ for the prime divisors $p$ of $n$. If there exists a prime $p$ that divides $n$ such that $n/p \equiv 2 \pmod 3$, Player 1 will win. If no such $p$ exists, but there is a $q$ that divides $n$ such that $n/q \equiv 1 \pmod 3$, Player 3 will win. Otherwise, Player 2 will win.*

**Nilpotent Groups**

**Theorem 5.3.** *In a game of Three-Player GEN on a nilpotent group $N$ with a Sylow 3-subgroup $P_3$, if it is the case that $3 \mid |N|$:*

- *Player 1 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 2 (mod\ 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*
- *Player 3 will win if $d(N) = 2$, $\dfrac{|N|}{9} \equiv 1 (mod\ 3)$ and $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*
- *Player 2 will win otherwise.*

**Theorem 5.4.** *The winner of the game of GEN on a nilpotent group $N$ such that $d(N) = 2$ can be determined by the value of $|N|$, with the exception of when $N$ has a Sylow 3-group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Specifically:*

- *If $|N| \equiv 1 (mod\ 3)$, Player 3 wins.*
- *If $|N| \equiv 2 (mod\ 3)$, Player 1 wins.*
- *If $|N| \equiv 0 (mod\ 3)$ and $P_3 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$, Player 2 wins.*
- *If $|N| \equiv 0 (mod\ 3)$, $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\dfrac{|N|}{9} \equiv 1 (mod\ 3)$ Player 3 wins.*
- *If $|N| \equiv 0 (mod\ 3)$, $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and $\dfrac{|N|}{9} \equiv 2 (mod\ 3)$ Player 1 wins.*

Moving forward, the challenge for any student researching Three-Player GEN will be tackling more general cases in which $d(G) \geq 3$. Obviously, these cases will bring a whole lot of uncertainty into the mix. In a game the depends so heavily on the final two generators, requiring three or more generators to make the group will drastically complicate the strategy.

By altering the rules of the game, students researching this topic will also be able to tackle new problems. What would these strategies look like if players would prefer that the person after them win instead of the person in front of them? Will the strategies remain very similar, or will the altered rule cause the game to change drastically? What would happen if a non-prime number of players, such as 4 or 6, played the game? Is there a generalization that can be made about playing with a prime number of players? These questions are waiting to be answered.

# Chapter 6

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