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Resonant solutions of the non-linear Schrödinger equation with periodic potential*

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Abstract
The goal is construction of stationary solutions close to non-trivial combinations of two plane waves at high energies for a periodic non-linear Schrödinger Equation in dimension two. The corresponding isoenergetic surface is described for any sufficiently large energy k^2. It is shown that the isoenergetic surface corresponding to k^2 is essentially different from that for the zero potential even for small potentials. We use a combination of the perturbative results obtained earlier for the linear case and a method of successive approximation.

Keywords: non-linear Schrödinger equation, periodic potential, dimension two, resonant, eigenfunctions

Mathematics Subject Classification numbers: 35Q55

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1. Introduction

Non-linear Schrödinger equation

$$\mathrm{i}u_t = -\Delta u + Vu + \sigma |u|^2 u \tag{1}$$

describes a variety of physical phenomena in optics, acoustics, quantum condensate (Gross-Pitaevskii equation), hydrodynamics, plasma physics, etc. The case of a periodic potential V is of great interest. The equation has been studied for a long time. However, majority of the studies is in physics literature and concerns with one-dimensional situation. Higher dimensions are investigated mostly numerically, (e.g. [1–5]) for dimension two). Theoretical papers for periodic multidimensional situations are more sparse. We definitely have to refer here to [6–13] devoted to periodic initial value problems with V being zero or an operator of multiplication in the Fourier dual space. We are interested in stationary solutions of (1) in multiple dimensions. In [14] the existence of stationary solutions decaying at infinity is investigated. It was proven in [15, 16] that there are stationary solutions of (1) close to plane waves $e^{i\langle \vec{k},\vec{x}\rangle}$ for an extensive set of $\vec{k}$ in dimensions two and three. In this paper we show that there exist stationary solutions close to non-trivial combinations of two plane waves at high energies for a periodic non-linear Schroedinger Equation in dimension two. The corresponding isoenergetic surfaces are described any sufficiently large energy k^2 . It is shown that the isoenergetic surface corresponding to k^2 is essentially different from that for the zero potential.

We start with considering a nonlinear polyharmonic equation:

$$(-\Delta)^l u(\vec{x}) + V(\vec{x})u(\vec{x}) + \sigma |u(\vec{x})|^2 u(\vec{x}) = \lambda u(\vec{x}),\ \vec{x} \in \mathbb{R}^n, \tag{2}$$

and the quasi-periodic boundary conditions:

$$\begin{cases} u(x_1,\dots,x_s+2\pi,\dots,x_n) = e^{2\pi\, \mathrm{i} t_s} u(x_1,\dots,x_s,\dots,x_n), \\ \frac{\partial}{\partial x_s} u(x_1,\dots,x_s+2\pi,\dots,x_n) = e^{2\pi\, \mathrm{i} t_s} \frac{\partial}{\partial x_s} u(x_1,\dots,x_s,\dots,x_n), \\ \vdots \\ \frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1,\dots,x_s+2\pi,\dots,x_n) = e^{2\pi\, \mathrm{i} t_s} \frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1,\dots,x_s,\dots,x_n), \\ s = 1,\dots,n. \end{cases} \tag{3}$$

where l is an integer such that $2l > n$ or $l = 1, n = 2$, $\sigma \in \mathbb{R}$ and $\vec{t} \in K$, $K = [0,1)^n$. We consider a periodic potential $V(\vec{x})$. We assume that the potential V is a trigonometric polynomial with the elementary cell of periods $Q = [0,2\pi]^n$:

$$V(\vec{x}) = \sum_{q \neq 0, |q| \leqslant R_0} v_q e^{\mathrm{i}\langle q,\vec{x}\rangle}, \quad 0 < R_0 < \infty, \tag{4}$$

$$\int_Q V(\vec{x})\, d\vec{x} = 0.$$

The last assumption can be done without loss of generality. We require that V is not identically zero. We consider the case $l = 1$, $n = 2$ for a sufficiently small $V : \|V\| < \varepsilon_*$, here $\varepsilon_* \neq \varepsilon_*(\lambda)$.

We start with discussing the linear operator:

$$H = (-\Delta)^l + V(\vec{x}) \tag{5}$$

in $L^2(\mathbb{R}^n)$. The spectral analysis of the operator H can be reduced to the analysis of operators $H(\vec{t}), \vec{t} \in K$. The operators $H(\vec{t})$ are defined by (5) and the quasiperiodic conditions (3) in $L^2(Q)$.

It is well-known that $H(\vec{t})$ has a discrete semi-bounded from below spectrum $\bigcup_{n=1}^{\infty} \lambda_n(\vec{t})$. By Bloch's theorem [17], the following is true:

1. The spectrum of H has band structure:

$$\Delta = \bigcup_{n=1}^{\infty} \bigcup_{\vec{t} \in K} \lambda_n\left(\vec{t}\right).$$

2. A complete system of generalized eigenfunctions of H can be obtained by extending eigenfunctions of $H(\vec{t})$ quasiperiodically to the whole space $\mathbb{R}^n$.

We use the notation $H_0(\vec{t})$ for $H(\vec{t})$ with $V = 0$, its eigenvalues being given by

$$p_j^{2l}\left(\vec{t}\right) = |\vec{P}_j\left(\vec{t}\right)|^{2l}, \tag{6}$$

where

$$\vec{P}_j\left(\vec{t}\right) = \vec{P}_j\left(\vec{0}\right) + \vec{t} = 2\pi j + \vec{t},\ j \in \mathbb{Z}^n,\ \vec{t} \in K.$$

The corresponding eigenfunction is a plane wave

$$\mathrm{e}^{\mathrm{i}\langle \vec{P}_j(\vec{t}), \vec{x}\rangle} \tag{7}$$

and the corresponding spectral projection is E_j defined by

$$\left(E_j\right)_{rm} = \delta_{jr}\delta_{jm} \tag{8}$$

in the basis (7) in $L^2(Q)$. Obviously any $\vec{k} \in \mathbb{R}^n$ can be uniquely written in the form $\vec{k} = \vec{P}_j(\vec{t})$ for some $j \in \mathbb{Z}^n$ and $\vec{t} \in K$. An isoenergetic surface $S_0(k)$ of H_0 is a set of $t \in K$ such that $p_j(t) = k$ for some $j \in Z^n$. It looks like a sphere 'packed' into K. Namely,

$$S_0 = \mathcal{K}S(k), \tag{9}$$

where $S(k)$ is the sphere in $\mathbb{R}^n$ centered at the origin with radius k and

$$\mathcal{K}: \mathbb{R}^n \to K,\ \ \mathcal{K}\vec{P}_q\left(\vec{t}\right) = \vec{t}. \tag{10}$$

The process of obtaining $S_0(k)$ starts by dividing $S(k)$ by the dual lattice $\{\vec{P}_m(0)\}_{m\in\mathbb{Z}^n}$ into pieces and all the pieces are then translated into K.

Perturbation series for eigenvalues and their spectral projections for $H(t)$ with respect to the operator $H_0(\vec{t})$ are obtained in [18]. When $2l > n$, perturbation formulas are valid for large k and a set $\chi_0(k,\delta) \subset S_0(k)$ of $\vec{t}$, such that $p_j(\vec{t}) = k$ for some j and

$$\min_{q\in\mathbb{Z}^n\setminus\{0\}} |p_j^{2l}\left(\vec{t}\right) - p_{j+q}^{2l}\left(\vec{t}\right)| > k^{2l-n-\delta}, \tag{11}$$

$0 < \delta < 2l - n$. For $n = 2,\ l = 1$, some additional inequalities are needed. This situation is called the non-resonant case for t. Correspondingly, $\chi_0(k,\delta)$ is the non-resonant set. The inequality (11) means that $\vec{t}$ is sufficiently far from self-intersections of $S_0(k)$. The set $\chi_0(k,\delta)$

has an asymptotically full measure on the isoenergetic surface $S_0(k)$ as $k \to \infty$. It is proven that for every $\vec{t} \in \chi_0(k,\delta)$, the operator $H(t)$ has an eigenvalue close to $p_j^{2l}(\vec{t}) = |\vec{P}_j(\vec{t})|^{2l}$ with an eigenfunction close to $e^{i\langle \vec{P}_j(\vec{t}),\vec{x}\rangle}$. The exact formulas of the eigenvalue and its spectral projection are given in section 2, theorem 3. In [15] we proved an analog of theorem 3 for the non-linear case (2) and (3); see theorem 14.

The vector $\vec{k}$ is said to satisfy the von Laue Diffraction Condition if

$$|\vec{k}| = |\vec{k} - \vec{P}_q\left(\vec{0}\right)|, \tag{12}$$

for some $q \in \mathbb{Z}^n \setminus \{0\}$. If $\vec{t} : \vec{P}_j(\vec{t}) = \vec{k}$, then, $\vec{t}$ obviously belongs to a self-intersection of $S_0(k)$, Therefore, $\vec{t} \notin \chi_0(k,\delta)$. Perturbation formulas for eigenvalues and spectral projections of $H(\vec{t})$ with respect to $H_0(\vec{t})$ do not work in this situation. The situation when $\vec{k}$ is in a vicinity of (12) is called the resonant case. It turns out that in a vicinity of (12) the eigenvalues and its spectral projections can be approximated by those of a model matrix

$$\begin{pmatrix} p_j^{2l}\left(\vec{t}\right) & v_q \\ v_{-q} & p_{j-q}^{2l}\left(\vec{t}\right) \end{pmatrix}, \tag{13}$$

where v_q is a Fourier coefficient of $V(x)$. Further, we assume:

$$q : v_q \neq 0. \tag{14}$$

We denote eigenvalues and eigenvectors of (13) by $\hat{\lambda}^{\pm}$ and $\hat{e}^{\pm}$, correspondingly. For a fixed λ_0 we consider the surface:

$$\hat{S}\left(\lambda_0\right) = \left\{\vec{t} \in K : \hat{\lambda}^{+}\left(\vec{t}\right) = \lambda_0. \text{ or } \hat{\lambda}^{-}\left(\vec{t}\right) = \lambda_0\right\}. \tag{15}$$

We call it the isoenergetic surface of the matrix (13). Note that the parts $\hat{\lambda}^{+}(\vec{t}) = \lambda_0$ and $\hat{\lambda}^{-}(\vec{t}) = \lambda_0$ do not intersect, since $v_q \neq 0$. Thus the deviation of the surface $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$ from the unperturbed one ($V = 0$) is essential.

In [18] we described a set $\chi_q(k,\delta) \subset S_0(k) \setminus \chi_0(k,\delta)$. Formulas for eigenvalues and spectral projections of $H(\vec{t})$ were constructed for every $\vec{t}$ in this set, when k is sufficiently large. Indeed, let set $\mathcal{K}S_q(k,n-2+\delta)$ be the image of the spherical layer $S_q(k,n-2+\delta)$ under the map $\mathcal{K}$ given by (10). Here,

$$S_q\left(k,n-2+\delta\right) = \left\{\vec{x} \in S\left(k\right) : \left||\vec{x}|^2 - |\vec{x} - \vec{P}_q\left(0\right)|^2\right| < 4k^{-n+2-\delta}\right\}. \tag{16}$$

In other words, we consider $S_q(k,n-2+\delta)$ being a neighborhood of (12) in the sphere $S(k)$ and then shift it into the cube K using (10). In section 2 here, we explicitly describe the set $\chi_q(k,\delta)$ which has an asymptotically full measure on $\mathcal{K}S_q(k,n-2+\delta)$. For $\vec{t}$ in a small vicinity of $\chi_q(k,\delta)$ we give formulas for eigenvalues and spectral projections of $H(\vec{t})$. The corresponding eigenfunctions are close to non-trivial combinations of two plane waves $e^{i<\vec{k},\vec{x}>}$ and $e^{i<\vec{k}-\vec{P}_q(\vec{0}),\vec{x}>}$. The coefficients of plane waves are described through eigenvectors $\hat{e}^{\pm}$ of (13). Thus, we have a pair of eigenfunctions for every $\vec{k}$ and a fixed q. The set $\chi_q(k,\delta)$ is called the resonant set. The exact statement of the result is given in section 2, theorem 9. It is proven that the isoenergetic surface corresponding to the pair of solutions is close to the isoenergetic surface (15), see corollary 13.

Here, we will construct formulas for solutions $u^{\pm}, \lambda^{\pm}$ of (2) and (3) when $\vec{k}$ is close to (12). Essentially, we make use of the results obtained for the linear case, and apply a method of successive approximation. Namely, we consider the part of equation (2) with the nonlinear term, i.e. $V + \sigma|u^{\pm}|^2$, as an unknown periodic potential. For the method of successive approximation, we define two maps, $\hat{\mathcal{M}}^{\pm}$, and construct two sequences of potentials, $W_m^{\pm}$ converging to the potentials $W^{\pm} = V + \sigma|u^{\pm}|^2$.

The map $\hat{\mathcal{M}}^{\pm}$ is defined in section 3.1. Then, the sequence of potentials $\{W_m^{\pm}\}_{m=0}^{\infty}$ is constructed as follows:

$$W_0^{\pm} = V, \tag{17}$$

$$.W_m^{\pm} = \hat{\mathcal{M}}^{\pm} W_{m-1}^{\pm}. \tag{18}$$

It turns out that it is a Cauchy sequence converging to a function W with respect to a norm $\|\cdot\|_*$,

$$\|W\|_* = \sum_{j\in\mathbb{Z}^n} |w_j|, \tag{19}$$

where w_j's are the Fourier coefficients of W. Namely, we show that

$$\|W^{\pm} - W_m^{\pm}\|_* \leqslant \left(ck^{-\gamma}\right)^{m+1}, \tag{20}$$

for some $\gamma > 0$. Then, we show that convergence of $\{W_m^{\pm}\}_{m=0}^{\infty}$ to $W^{\pm}$ leads to convergence of the sequences of the spectral projections $\{\hat{E}_m^{\pm}(\vec{t})\}_{m=0}^{\infty}$ and their corresponding eigenvalues $\{\hat{\lambda}_m^{\pm}(\vec{t})\}_{m=0}^{\infty}$ to $\hat{E}_W^{\pm}(\vec{t})$ and $\hat{\lambda}_W^{\pm}(\vec{t})$, here $\hat{E}_m^{\pm}(\vec{t})$, $\hat{\lambda}_m^{\pm}(\vec{t})$ and $\hat{E}_W^{\pm}(\vec{t})$, $\hat{\lambda}_W^{\pm}(\vec{t})$ are spectral projections and their eigenvalues of $H_0 + W_m^{\pm}$ and $H_0 + W^{\pm}$. Corresponding to $\hat{E}_W^{\pm}(\vec{t})$ eigenfunction $u^{\pm} \equiv \hat{u}_W^{\pm}$, see (72) and (73), solves (2) and (3). It is shown that $u^{\pm}$ is close to a combination of two plane waves $\mathrm{e}^{\mathrm{i}<\vec{k},\vec{x}>}$ and $\mathrm{e}^{\mathrm{i}<\vec{k}-\vec{P}_q(\vec{0}),\vec{x}>}$ under some restriction on its amplitude. The coefficients of plane waves are described by eigenvectors $\hat{e}^{\pm}$ of (13) with a good accuracy. The exact statement of the result is given by theorem 24. The sequences $\{W_m^{\pm}\}_{m=0}^{\infty}$, $\{\hat{E}_m^{\pm}(\vec{t})\}_{m=0}^{\infty}$ and $\{\hat{\lambda}_m^{\pm}(\vec{t})\}_{m=0}^{\infty}$. can be differentiated with respect to $\vec{t}$ and maintain their convergence. As a result the asymptotic formulas (141) and (142) for $\nabla\hat{E}_W^{\pm}(\vec{t})$, $\nabla\hat{\lambda}_W^{\pm}(\vec{t})$ are proven, and $|\nabla\hat{\lambda}_W^{\pm}(\vec{t})| \approx 2lk^{2l-1}$. It follows that the surface $\lambda^{\pm}(\vec{t}) = \lambda_0$ is in the $C(V)\lambda_0^{-\hat{\gamma}}$-neighborhood of $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$ for every sufficiently large λ_0, here $\hat{\lambda}^{\pm}$ are eigenvalues of (13) and $\hat{\gamma} > (2l-1)(2l)^{-1}$, see theorem 30 and corollary 31.

In section 4 we consider the physically interesting case $l = 1$, $n = 2$. All considerations of the previous case can be done for $l = 1$, $n = 2$ under the assumption that V is sufficiently small $\|V\|_* < \epsilon^9$, $0 < \epsilon < \epsilon_0$, $\epsilon_0 \neq \epsilon_0(\lambda)$, for the waves with a sufficiently small amplitude A, the restriction on A being given in terms of ϵ, see (150).

The isoenergetic surface $\lambda^{\pm}(\vec{t}) = \lambda_0$ is in a small neighborhood of $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$.

Thus, we prove that for a relatively small set of momenta $\vec{t}$ there are solutions $u^{\pm}(\vec{x})$ of the problem (2) and (3), neither of them being close to a plane wave even for a small V. This phenomena is important in Physics of Solids already for a linear case ($\sigma = 0$). It describes a reflection of a beam of electrons in a crystal lattice. Measuring the conditions on $\vec{k}$ for such a reflection (a small set of $\vec{k}$), physicists define a lattice type of the crystal. The phenomena made it possible to develop the field of electron crystallography. We show here that the phenomena persists for a non-linear case even for small potentials.

In our considerations the solutions $u^{\pm}$, $\lambda^{\pm}$ and the isoenergetic surfaces depend not only on the periods of $V(\vec{x})$, but also on its Fourier coefficients. This may look to be in disagreement with the von Laue diffraction condition (12) and other related formulas, which is known to physicists since long time ago. In those formulas there are no dependence on Fourier coefficients. The reason for this 'difference' is that von Laue diffraction conditions can be detected with the type of equipment that existed for many decades. The newer equipment can detect more complicated diffraction picture. This enables researches to determine the molecules constituting the crystal, i.e. some important features of $V(x)$.

In the present case we have only two waves $\mathrm{e}^{\mathrm{i}\langle\vec{P}_j(\vec{t}),\vec{x}\rangle}$ and $\mathrm{e}^{\mathrm{i}\langle\vec{P}_{j-q}(\vec{t}),\vec{x}\rangle}$ with significant amplitudes in the Fourier expansions of $u^{\pm}(\vec{x})$. It is due to the fact that $|\vec{P}_{j-q}(\vec{t})|^{2l}$ is close to $|\vec{P}_j(\vec{t})|^{2l}$, while all other $|\vec{P}_{j-m}(\vec{t})|^{2l}$, $m\neq 0,q$ are relatively far away from the pair, when $\vec{t}\in\chi_q(k,\delta)$. More precisely, $||\vec{P}_{j-q}(\vec{t})|^2-|\vec{P}_j(\vec{t})|^2|<c\|V\|$, c being the absolute constant, while $|\vec{P}_{j-m}(\vec{t})|^{2l}$, $m\neq 0,q$ satisfy the opposite inequality. Generally speaking, the number of plane waves $\mathrm{e}^{\mathrm{i}\langle\vec{P}_{j-q}(\vec{t}),\vec{x}\rangle}$, $q\in\mathbb{Z}^n$ with significant amplitudes in the Fourier expansions of $u^{\pm}(\vec{x})$ is defined by the number of real points $|\vec{P}_{j-q}(\vec{t})|^{2l}$, $q\in Z^n\setminus\{0\}$ in a $(c\|V\|)$-vicinity of $k^2=|\vec{P}_j(\vec{t})|^{2l}$. Therefore, more than two waves are involved if we have more than one point $|\vec{P}_{j-q}(\vec{t})|^{2l}$ in such a vicinity (in the present case we have just one). In particular, if $n=2$, $l=1$ and $\|V\|$ is sufficiently large, then all plane waves $\mathrm{e}^{\mathrm{i}\langle\vec{P}_{j-q}(\vec{t}),\vec{x}\rangle}$, $q:||\vec{P}_{j-q}(\vec{t})|^2-|\vec{P}_j(\vec{t})|^2|<c\|V\|$ play significant roles. In this case the model operator is described not by (13), but by a Hill operator. This result will be proven in a forthcoming paper.

The paper is organized as follows. Preliminary results are described in sections 2 and 2.1 containing the results for a linear case [18] and section 2.2 containing the results for the nonlinear equation, non-resonant case [15]. Chapter 3 is devoted to proving the main result for the case $2l>n$. We consider $l=1,n=2$ in chapter 4.

2. Preliminary results

In this chapter, we present a brief review of previous results needed for proof of the main result of this paper.

2.1. Linear polyharmonic equation with periodic potential

Let us consider an operator in $L_2(Q)$ given by the differential expression:

$$H_\alpha\left(\vec{t}\right)u=(-\Delta)^l u+\alpha V u, \tag{21}$$

with the quasi-periodic boundary conditions (3). Here l is an integer such that $2l>n$, $-1\leqslant\alpha\leqslant 1$.

Perturbation series for eigenvalues and its spectral projections for $H_\alpha(t)$ are constructed on a nonsingular set χ_0 [18]. We include here a construction of $\chi_0(k,\delta)$ and discuss the perturbation theory for $H_\alpha(\vec{t})$.

Lemma 1. *For an arbitrarily small positive, δ, $2\delta<2l-n$, and sufficiently large $k>k_0(\delta)$, there exists a non-resonant set $\chi_0(k,\delta)$, belonging to the isoenergetic surface $S_0(k)$ of the free operator $H_0(t)$, such that, for any $\vec{t}$ in it,*

$$1.\ \text{there exists a unique} m\in\mathbb{Z}^n \text{such that} p_m^{2l}\left(\vec{t}\right)=k^{2l}, \tag{22}$$

$$2.\ \min_{j\neq m}|p_j^{2l}\left(\vec{t}\right)-p_m^{2l}\left(\vec{t}\right)|>k^{2l-n-\delta}. \tag{23}$$

Moreover, the nonsingular set has an asymptotically full measure on $S_0(k)$:

$$\frac{s\left(S_0\left(k\right)\setminus\chi_0\left(k,\delta\right)\right)}{s\left(S_0\left(k\right)\right)}=O\left(k^{-\delta/2}\right),\ as\ k\to\infty,$$

where $s(.)$ denotes the Lebesgue measure.

Corollary 2. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the resonant set $\chi_0(k,\delta)$, $0<2\delta<2l-n$. Then for all z lying on the circle $C_0=\{z\in\mathbb{C}:|z-k^{2l}|=k^{2l-n-\delta}\}$ and any i of $\mathbb{Z}^n$, the inequality*

$$2\mid p_i^{2l}\left(\vec{t}\right)-z\mid>k^{2l-n-\delta}$$

holds.

Let us introduce the functions $g_r(k,\vec{t})$ and the operator-valued functions $G_r(k,\vec{t})$, $r=0$, 1, …, for $\vec{t}\in\chi_0(k,\delta)$:

$$g_r\left(k,\vec{t}\right)=\frac{(-1)^r}{2\pi ir}\mathrm{Tr}\oint_{C_0}\left(\left(H_0\left(\vec{t}\right)-z\right)^{-1}V\right)^r\mathrm{d}z, \tag{24}$$

$$G_r\left(k,\vec{t}\right)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_0}\left(\left(H_0\left(\vec{t}\right)-z\right)^{-1}V\right)^r\left(H_0\left(\vec{t}\right)-z\right)^{-1}\mathrm{d}z. \tag{25}$$

The following theorem presents the main result for (21).

Theorem 3. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the non-resonant set $\chi_0(k,\delta)$, $0<2\delta<2l-n$. Then for sufficiently large k, $k>k_0(\|V\|,\delta)$, there exists a single eigenvalue of the operator $H(\vec{t})$ in the interval $\varepsilon(k,\delta)\equiv(k^{2l}-k^{2l-n-\delta},k^{2l}+k^{2l-n-\delta})$. It is given by the series*

$$\lambda_j\left(\alpha,\vec{t}\right)=p_j^{2l}\left(\vec{t}\right)+\sum_{r=2}^{\infty}\alpha^r g_r\left(k,\vec{t}\right), \tag{26}$$

converging absolutely in the disk $|\alpha|\leqslant1$, where the index j is uniquely determined from the relation $p_j^{2l}(\vec{t})\in\varepsilon(k,\delta)$ and the spectral projection, corresponding to $\lambda_j(\alpha,\vec{t})$ is given by the series

$$E_j\left(\alpha,\vec{t}\right)=E_j+\sum_{r=1}^{\infty}\alpha^r G_r\left(k,\vec{t}\right), \tag{27}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\leqslant1$.

Moreover, for the coefficients $g_r(k,\vec{t})$ and $G_r(k,\vec{t})$, the following estimates hold.

$$|g_r\left(k,\vec{t}\right)|<k^{2l-n-\delta}k^{-(2l-n-2\delta)r}, \tag{28}$$

$$\|G_r\left(k,\vec{t}\right)\|_1\leqslant k^{-(2l-n-2\delta)r}. \tag{29}$$

The series (26) and (27) are differentiable termwise with respect to $\vec{t}$ in the $(k^{-n+1-2\delta})$-neighborhood in $\mathbb{C}^n$ of the set $\chi_0(k,\delta)$, see [18]. Indeed, let

$$T(m)\equiv\frac{\partial^{|m|}}{\partial t_1^{m_1}\partial t_2^{m_2}\dots\partial t_n^{m_n}},\text{where }|\,m\,|\equiv m_1+m_2+\ldots+m_n, \tag{30}$$

$$m!\equiv m_1!m_2!\dots m_n!,\quad 0\leqslant|\,m\,|<\infty,\quad T(0)f\equiv f. \tag{31}$$

Theorem 4. *Under the conditions of theorem 3 the series (26) and (27) can be differentiated with respect to $\vec{t}$ any number of times, and they retain their asymptotic character. The coefficients $g_r(k,\vec{t})$ and $G_r(k,\vec{t})$ satisfy the following estimates in the $(k^{-n+1-2\delta})$-neighborhood in $\mathbb{C}^n$ of the nonsingular set $\chi_0(k,\delta)$:*

$$|\, T(m)\, g_r\left(k,\vec{t}\right) | < m! k^{-(2l-n-2\delta)(r-1)} k^{|m|(n-1+2\delta)} \tag{32}$$

$$\|T(m)\, G_r\left(k,\vec{t}\right)\|_1 < m! k^{-(2l-n-2\delta)r} k^{|m|(n-1+2\delta)}. \tag{33}$$

Corollary 5. *The following estimates hold for the perturbed eigenvalue and its spectral projection.*

$$|\, T(m)\left(\lambda_j\left(\alpha,\vec{t}\right) - p_j^{2l}\left(\vec{t}\right)\right) | < c m! k^{-(2l-n-2\delta)} k^{|m|(n-1+2\delta)} \tag{34}$$

$$\|T(m)\left(E_j\left(\alpha,\vec{t}\right) - E_j\right)\|_1 < c m! k^{-(2l-n-2\delta)} k^{|m|(n-1+2\delta)}. \tag{35}$$

In particular,

$$|\, \lambda_j(\alpha,\vec{t})) - p_j^{2l}(\vec{t}) \,| < c k^{-(2l-n-2\delta)} \tag{36}$$

$$\|E_j(\alpha,\vec{t}) - E_j)\|_1 < c k^{-(2l-n-2\delta)} \tag{37}$$

$$|\, \nabla_{\vec{t}} \lambda_j\left(\alpha,\vec{t}\right) - 2l\vec{P}_j\left(\vec{t}\right) p_j^{2l-2}\left(\vec{t}\right) | < c k^{-(2l-n-2\delta)+n-1+2\delta}. \tag{38}$$

Corollary 6. *The surface $\lambda_j(\alpha,\vec{t}) = \lambda_0$ is in the real $\lambda_0^{-(4l-n+1-2\delta)}$-neighborhood of $\chi_0(k,\delta)$ for every sufficiently large λ_0.*

Next, we consider formulas for $\vec{t} \in S_0(k) \setminus \chi_0(k,\delta)$. This means that there is at least one $q \in \mathbb{Z}^n$ such that

$$|p_{j-q}^{2l}\left(\vec{t}\right) - p_j^{2l}\left(\vec{t}\right)| < k^{2l-n-\delta},\ \vec{P}_j\left(\vec{t}\right) \in S(k).$$

Below, we present the main results and the perturbation formulas for an eigenvalue and its spectral projection for $\vec{t}$ belonging to a 'resonant set', $\chi_q(k,\delta) \subset S_0(k) \setminus \chi_0(k,\delta)$. We set $\alpha = 1$ for simplicity.

Let P_q be the diagonal operator in $l^2(\mathbb{Z}^n)$ defined by the formula

$$\left(P_q\right)_{mm} = \delta_{jm} + \delta_{j-q,m}. \tag{39}$$

We define the operator $\hat{H}_q(\vec{t})$ as

$$\hat{H}_q\left(\vec{t}\right) = H_0\left(\vec{t}\right) + P_q V P_q. \tag{40}$$

Note that the matrix corresponding to this operator has only two non-diagonal elements, namely,

$$\hat{H}_q\left(\vec{t}\right)_{j,j-q} = \overline{\hat{H}_q\left(\vec{t}\right)_{j-q,j}} = v_q.$$

Thus $\hat{H}_q(\vec{t})$ has only one block (13), here we assume (14). The eigenvalues of this block matrix we denote by $\hat{\lambda}^+(\vec{t})$ and $\hat{\lambda}^-(\vec{t})$. They are given as

$$\hat{\lambda}^+\left(\vec{t}\right) = a + b \text{ and } \hat{\lambda}^-\left(\vec{t}\right) = a - b, \tag{41}$$

where $2a = p_j^{2l}(\vec{t}) + p_{j-q}^{2l}(\vec{t})$ and $2b = (4|v_q|^2 + (p_j^{2l}(\vec{t}) - p_{j-q}^{2l}(\vec{t}))^2)^{1/2}$. Note that

$$\hat{\lambda}^+(\vec{t}) - \hat{\lambda}^-(\vec{t}) = 2b \geqslant 2|v_q|.$$

This means that the spectrum of $\hat{H}_q(\vec{t})$ is $\{\{p_i^{2l}(\vec{t})\}_{i\neq j,j-q}, \hat{\lambda}^+(\vec{t}), \hat{\lambda}^-(\vec{t})\}$.

Definition 1. The spectral projections of $\hat{H}_q(\vec{t})$ corresponding to $\hat{\lambda}^{\pm}(\vec{t})$ we denote by $= \hat{E}^{\pm}$.

Recall that $\mathcal{K}S_q(k, n-2+\delta)$ was defined in section 1, see (10) and (16).

Lemma 7. *For an arbitrarily small positive δ, $2\delta < 2l-n$, and sufficiently large k, there exists a subset $\chi_q(k,\delta)$ of $\mathcal{K}S_q(k,n-2+\delta)$, such that the following conditions hold*

1. *There exists $j \in \mathbb{Z}^n$ such that $\vec{P}_j(\vec{t}) = \vec{k}$,*
2. $|p_{j-q}^2(\vec{t}) - p_j^2(\vec{t})| < k^{-n+2-\delta}$,
3. $\min\limits_{i\neq j,j-q} |p_j^2(\vec{t}) - p_i^2(\vec{t})| > 2k^{-n+2-6\delta}$.

for all $\vec{t} \in \chi_q(k,\delta)$. Moreover, for any $\vec{t}$ in the k-neighborhood of $\chi_q(k,\delta)$ in $\mathbb{C}^n$, there exists a unique $j \in \mathbb{Z}^n$ such that $|p_j^2(\vec{t}) - k^2| < 5k^{-n+2-6\delta}$ and the second and third conditions above are satisfied. Also, the set $\chi_q(k,\delta)$ has an asymptotically full measure on $\mathcal{K}S_q(k,n-2+\delta)$, that is

$$\frac{s(\mathcal{K}S_q(k,n-2+\delta) \setminus \chi_q(k,\delta))}{s(\mathcal{K}S_q(k,n-2+\delta))} = O\left(k^{-\delta/2}\right) \quad \text{as } k \to \infty. \tag{42}$$

The previous lemma means that $p_j^2(\vec{t})$ and $p_{j-q}^2(\vec{t})$ are close to each other, but they are sufficiently far away from the remaining eigenvalues.

Corollary 8. *If $\vec{t}$ belongs to the $(2k^{-n+1-7\delta})$- neighborhood of $\chi_q(k,\delta)$, then for all z on the circle*

$$C_1^+ = \left\{z : |z - \hat{\lambda}^+(\vec{t})| = d\right\}, \quad d = \frac{1}{10}|v_q|, \tag{43}$$

both of the following inequalities are true:

$$2|p_m^{2l}(\vec{t}) - z| \geqslant k^{2l-n-6\delta}, m \neq j, j-q_0, \tag{44}$$

$$|\hat{\lambda}^-(\vec{t}) - z| \geqslant d. \tag{45}$$

Similar corollary holds for $\hat{\lambda}^-, C_1^-$. Further for definiteness we consider $\hat{\lambda}^+, C_1^+$.

Let V be as in (4) and functions $\hat{g}_r^+(k,\vec{t})$ and $\hat{G}_r^+(k,\vec{t})$, $r \in \mathbb{N}$, $t \in \chi_q(k,\delta)$ be defined as follows:

$$\hat{g}_r^+(k,\vec{t}) = \frac{(-1)^r}{2\pi i r} \mathrm{Tr} \oint_{C_1^+} \left(\left(\hat{H}_q(\vec{t}) - z\right)^{-1} \hat{V}\right)^r \mathrm{d}z, \tag{46}$$

$$\hat{G}_r^+(k,\vec{t}) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_1^+} \left(\left(\hat{H}_q(\vec{t}) - z\right)^{-1} \hat{V}\right)^r \left(\hat{H}_q(\vec{t}) - z\right)^{-1} \mathrm{d}z, \tag{47}$$

here and below

$$\hat{V} = V - P_q V P_q.$$

The following result is proven in [18].

Theorem 9. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-7\delta})$-neighborhood in K of the set $\chi_q(k,\delta)$, $0<9\delta<2l-n$. Then for sufficiently large k, $k>k_1(V,\delta)$, in the interval $\hat{\varepsilon}(k,\delta)\equiv(\hat{\lambda}^+(\vec{t})-k^{-\delta},\hat{\lambda}^+(\vec{t})+k^{-\delta})$, there exists a single eigenvalue of the operator $H(\vec{t})$. It is given by the series*

$$\hat{\lambda}^{q,+}\left(\vec{t}\right)=\hat{\lambda}^{+}\left(\vec{t}\right)+\sum_{r=2}^{\infty}\hat{g}_r^{+}\left(k,\vec{t}\right). \tag{48}$$

The spectral projection, corresponding to $\hat{\lambda}^q(\vec{t})$ is given by

$$\hat{E}^{q,+}\left(\vec{t}\right)=\hat{E}^{+}+\sum_{r=1}^{\infty}\hat{G}_r^{+}\left(k,\vec{t}\right), \tag{49}$$

which converges in the class $\mathbf{S_1}$. The following estimates hold for $\hat{g}_r^+(k,\vec{t})$, $\hat{G}_r^+(k,\vec{t})$:

$$|\hat{g}_r^{+}\left(k,\vec{t}\right)|<k^{-\gamma_1 r-\delta}, \tag{50}$$

$$\|\hat{G}_r^{+}\left(k,\vec{t}\right)\|_1<k^{-\gamma_1 r}, \tag{51}$$

where

$$\gamma_1=(2l-n)/2-4\delta>0. \tag{52}$$

Theorem 10. *Under the conditions of theorem 9, the series (48) and (49) can be differentiated termwise with respect to $\vec{t}$ any number of times, and they retain their asymptotic character. The coefficients $\tilde{g}_r(k,\vec{t})$ and $\tilde{G}_r(k,\vec{t})$ satisfy the following estimates in the $(k^{-n+1-7\delta})$-neighborhood in $\mathbb{C}^n$ of the singular set $\chi_q(k,\delta)$:*

$$|T(m)\,\hat{g}_r^{+}\left(k,\vec{t}\right)|<m!k^{-\gamma_1 r-\delta+|m|(n-1+7\delta)}, \tag{53}$$

$$\|T(m)\,\hat{G}_r^{+}\left(k,\vec{t}\right)\|_1<m!k^{-\gamma_1 r+|m|(n-1+7\delta)}. \tag{54}$$

Corollary 11. *The following estimates for the perturbed eigenvalue and its spectral projection hold:*

$$|T(m)\left(\hat{\lambda}^{q,+}\left(\vec{t}\right)-\hat{\lambda}^{+}\left(\vec{t}\right)\right)|<2m!k^{-2\gamma_1-\delta+|m|(n-1+7\delta)}, \tag{55}$$

$$\|T(m)\left(\hat{E}^{q,+}\left(\vec{t}\right)-\hat{E}^{+}\left(\vec{t}\right)\right)\|_1<2m!k^{-\gamma_1+|m|(n-1+7\delta)}. \tag{56}$$

In particular, the following estimates are valid:

$$|\hat{\lambda}^{q,+}\left(\vec{t}\right)-\hat{\lambda}^{+}\left(\vec{t}\right)|<2k^{-2\gamma_1-\delta}, \tag{57}$$

$$\|\hat{E}^{q,+}\left(\vec{t}\right)-\hat{E}^{+}\|_1<2k^{-\gamma_1}, \tag{58}$$

$$|\nabla\hat{\lambda}^{q,+}\left(\vec{t}\right)-\nabla\hat{\lambda}^{+}\left(\vec{t}\right)|<2k^{-2\gamma_1+n-1+6\delta}. \tag{59}$$

Corollary 12. *An eigenfunction $u_0^+(\vec{x},\vec{t})$ corresponding to $\hat{E}^{q,+}(\vec{t})$ satisfies the estimate:*

$$\left\| \mathrm{e}^{-\mathrm{i}<\vec{k},\vec{x}>} u_0^+\left(\vec{x},\vec{t}\right) - \left(a_1 + a_2 \mathrm{e}^{-\mathrm{i}<\vec{P}_q(0),\vec{x}>}\right) \right\|_* < ck^{-\gamma_1}, \tag{60}$$

where (a_1,a_2) is an eigenvector of (13) corresponding to $\hat{\lambda}^+(\vec{t})$.

The analogous results hold for $\hat{\lambda}^{q,-}(\vec{t}), \hat{E}^{q,-}(\vec{t})$.

Corollary 13. *The surface $\hat{\lambda}^{q,\pm}(\vec{t}) = \lambda_0$ is in the real $\lambda_0^{-2\gamma_1-2l+1-\delta}$-neighborhood of $\chi_q(k,\delta)$ for every sufficiently large λ_0.*

The results above hold for $V : \|V\|_* < \infty$.

2.2. Nonlinear periodic polyharmonic equation

In this section we consider equation (2) with the quasi-periodic boundary conditions (3). In [15, 16] we proved existence of a quasi-periodic solution of equation (2) being close to a plane wave $A\mathrm{e}^{\mathrm{i}\langle\vec{k},\vec{x}\rangle}$ for every $\vec{k}$ belonging to a non-resonant set $\chi_0(k)$, where A is a complex number with sufficiently small $|A|$. The main idea of the papers is to look at equation (2) as a linear equation. This can be done by considering the sum $V(\vec{x}) + \sigma|u(\vec{x})|^2$ as an unknown potential. We use the facts proved for the linear case.

Now we describe the technique used to find a function u that solves the nonlinear equation. Let $\hat{l}_1$ be the space of functions in Q with Fourier coefficients in $l_1(\mathbb{Z}^n)$, see (19). First, a sequence of operators $\{W_m\}_{m=0}^{\infty}$ is constructed via the map $\mathcal{M} : \hat{l}_1 \to \hat{l}_1$ defined as

$$\mathcal{M}W(\vec{x}) = V(\vec{x}) + \sigma|u_{\widetilde{W}}(\vec{x})|^2. \tag{61}$$

Here, $u_{\widetilde{W}}$ is an eigenfunction of the linear operator $(-\Delta)^l + \widetilde{W}$ with the same boundary conditions (3), where $\widetilde{W}$ is defined by the formula

$$\widetilde{W}(\vec{x}) = W(\vec{x}) - \frac{1}{(2\pi)^n}\int_Q W(\vec{x})\,\mathrm{d}\vec{x}. \tag{62}$$

More precisely, $u_{\widetilde{W}}$ is defined as

$$\begin{aligned} u_{\widetilde{W}}(\vec{x}) &= \sum_{s\in\mathbb{Z}^n} \left(E_{\widetilde{W}}(\vec{t})\right)_{sj} \mathrm{e}^{\mathrm{i}\langle\vec{P}_s(\vec{t}),\vec{x}\rangle} \\ &= \mathrm{e}^{\mathrm{i}\langle\vec{P}_j(\vec{t}),\vec{x}\rangle} + \sum_{r=1}^{\infty}\sum_{s\in\mathbb{Z}^n}\left(G^r_{\widetilde{W}}(k,\vec{t})\right)_{sj} \mathrm{e}^{\mathrm{i}\langle\vec{P}_s(\vec{t}),\vec{x}\rangle}, \end{aligned} \tag{63}$$

here $j : \vec{P}_j(\vec{t}) = \vec{k}$. Formula (63) is analogous to (27). Second, the operator W_m is defined by a recurrence procedure as follows. Let $W_0 = V + \sigma|A|^2$. Define

$$W_{m+1} = \mathcal{M}W_m,\ m = 0,1,2,\ldots \tag{64}$$

Then, W_m is proved to be a Cauchy sequence of periodic potentials converging to a periodic potential W with respect to the norm $\|\cdot\|_*$, see (19). Note that for any m, there is a solution u_m of the equation

$$(-\Delta)^l u_m + \widetilde{W}_m u_m = \lambda_m u_m,$$

given by the formula (63). Moreover, the sequences of the functions $\{u_m\}_{m=0}^{\infty}$ and the eigenvalues $\{\lambda_m\}_{m=0}^{\infty}$ converge to a function $u_{\widetilde{W}}$ and a real number λ, respectively. It is shown that $u = u_{\widetilde{W}}$ solves the nonlinear equation (2). The limits $E_{\widetilde{W}}(\vec{t})$ and $\lambda_{\widetilde{W}}(\vec{t})$ of the sequences $\{E_m(\vec{t})\}_{m=0}^{\infty}$ and $\{\lambda_m(\vec{t})\}_{m=0}^{\infty}$, respectively, are given by the two series (26) and (27) written for the potential $\widetilde{W}$ instead of V. The following theorem holds when $2l > n$.

Theorem 14. *Let $0 < 2\delta < 2l - n$. Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the non-resonant set $\chi_0(k,\delta)$, $k > k_1(\|V\|_*,\delta)$, and $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$, where*

$$\gamma_0 = 2l - n - 2\delta. \tag{65}$$

Then, there is a function u, depending on $\vec{t}$ as a parameter, and a real value $\lambda(\vec{t})$, such that they solve the equation (2) and the quasi-periodic boundary condition (3). The following estimates hold:

$$\lambda\left(\vec{t}\right) = p_j^{2l}\left(\vec{t}\right) + \sigma|A|^2 + O\left(k^{-\gamma_0}\right), \tag{66}$$

$$u\left(\vec{x}\right) = Ae^{\mathrm{i}\langle\vec{p}_j(\vec{t}),\vec{x}\rangle}\left(1 + \tilde{u}\left(\vec{x}\right)\right), \tag{67}$$

where $\tilde{u}$ is periodic and

$$\|\tilde{u}\|_* < k^{-\gamma_0}. \tag{68}$$

Moreover, the following estimates hold:

$$\left|\nabla\left(\lambda\left(\vec{t}\right) - p_j^{2l}\left(\vec{t}\right) - \sigma|A|^2\right)\right| < k^{2l-1-\gamma_0}. \tag{69}$$

Corollary 15. *The surface $\lambda(\vec{t}) = \lambda_0$, $\lambda_0 \equiv k_0^{2l} + \sigma|A|^2$, is in the $k_0^{-(4l-n+1-2\delta)}$-neighborhood of $\chi_0(k,\delta)$ for every fixed $A : |\sigma||A|^2 < k_0^{\gamma_0-\delta}$ and sufficiently large λ_0.*

If $l = 1, n = 2$ an analogous theorem holds with somewhat different non-resonant set set $\chi_0(k,\delta)$, and a constant γ_0.

3. Perturbation theory for non-linear polyharmonic equation with periodic potential for $2l > n$. Resonant case

In section 3.1, we introduce maps $\hat{\mathcal{M}}^{\pm}$ and construct two sequences of potentials $\{W_m^{\pm}\}_{m=0}^{\infty}$. We will prove that they are Cauchy sequences converging to some potentials $W^{\pm}$. In section 3.2 we prove the existence of solutions $u^{\pm}(\vec{t},\vec{x})$, $\lambda = \lambda^{\pm}(\vec{t})$ of (2) and (3) and obtain their estimates at the high energy region. The functions $u^{\pm}(\vec{t},\vec{x})$ are shown to be close to non-trivial combinations of two plane waves. Finally, in section 3.3, we derive estimates for the derivatives of $\lambda^{\pm}(\vec{t})$ and $u^{\pm}(\vec{t},\vec{x})$ with respect to $\vec{t}$. The isoenergetic surface $\{t \in K, \lambda^{\pm}(\vec{t}) = \lambda_0\}$ is constructed for the resonant case for every sufficiently large λ_0.

3.1. Construction of a Cauchy sequence

Recall that the operator $\hat{H}_q(\vec{t})$ defined by (40) has the spectrum $\{\{p_i^{2l}(\vec{t})\}_{i\neq j,j-q}, \hat{\lambda}^+(\vec{t}), \hat{\lambda}^-(\vec{t})\}$. It is easy to verify that eigenvectors of the block submatrix of $\hat{H}_q(\vec{t})$ corresponding to the eigenvalues $\hat{\lambda}^{\pm}(\vec{t})$, see (41), are:

$$\hat{e}^+ = \left(p_{j-q}^{2l}\left(\vec{t}\right) - \hat{\lambda}^+\left(\vec{t}\right), \ -v_q\right)^T \tag{70}$$

$$\hat{e}^- = \left(-v_{-q}, \ p_j^{2l}\left(\vec{t}\right) - \hat{\lambda}^-\left(\vec{t}\right)\right)^T. \tag{71}$$

By definition 1 the corresponding spectral projections are denoted by by $\hat{E}^{\pm}(\vec{t})$. It is clear that $|\hat{\lambda}^+(\vec{t}) - p_j^{2l}(\vec{t})| < 2|v_q|$ and $|\hat{\lambda}^-(\vec{t}) - p_j^{2l}(\vec{t})| < 2|v_q|$. We assume that $|v_q| \neq 0$.

The cases $+$ and $-$ are analogous. Further, for definiteness we consider the case $+$.

Wee use a geometric lemma 7 and corollary 8.

Definition 2. Let W be such that $\|W\|_* < \infty$, $\vec{t} \in \chi_q(k,\delta)$ and

$$u_W^+(\vec{x}) = \psi_W^+(\vec{x})\,\mathrm{e}^{\mathrm{i}\langle\vec{P}_j(\vec{t}),\vec{x}\rangle}, \tag{72}$$

where ψ_W^+ is periodic and

$$\psi_W^+(\vec{x}) = A\sum_{b\in\mathbb{Z}^n}\hat{E}_W^{q,+}\left(\vec{t}\right)_{j-b,j}\mathrm{e}^{\mathrm{i}\langle\vec{P}_b(0),\vec{x}\rangle} \tag{73}$$

$\hat{E}_W^{q,+}$ being given by (47) and (49) with $W - P_qVP_q$ in (47) instead of $\hat{V}$.

Obviously, $u_W^+(x)$ satisfies $(-\Delta)^l u_W^+ + Wu_W^+ = \lambda_W^+(t)u_W^+$, where $\lambda_W^+(t)$ is given by (48) with $W - P_qVP_q$ in (47) instead of $\hat{V}$.

Definition 3. Let the map $\hat{\mathcal{M}}^+ : \hat{l}_1 \to \hat{l}_1$ be defined by

$$\hat{\mathcal{M}}^+W(\vec{x}) = V(\vec{x}) + \sigma|u_W^+(\vec{x})|^2, \tag{74}$$

here $\hat{l}_1$ is the space of functions in Q with Fourier coefficients in $l_1(\mathbb{Z}^n)$, see (19).

Next, we define the sequence $\{W_m^+\}_{m=0}^{\infty}$ using the map $\hat{\mathcal{M}}^+$. Let

$$W_0^+ = V, \tag{75}$$

$$W_m^+ = \hat{\mathcal{M}}^+W_{m-1}^+. \tag{76}$$

For every m there is an eigenfunction $u_m^+(\vec{x})$ corresponding W_m^+, see (72), and the corresponding eigenvalue number λ_m^+. Note that this sequence W_m^+ is quite different from that for a non-resonance case given by (64).

Definition 4. Let, $\hat{E}_{-1}^+ \equiv \hat{E}^+$ be the spectral projection of the model operator $\hat{H}_q$ corresponding to $\lambda^+(\vec{t})$, and $\hat{E}_0^+ = \hat{E}^{q,+}$, see (49). By analogy, $\hat{E}_m^+ \equiv \hat{E}_{W_m^+}^{q,+}(\vec{t})$, the formula (47) and (49) being used for $E_{W_m^+}^+(\vec{t})$ with $W_m^+ - P_qVP_q$ instead of $\hat{V}$.

Clearly, $\hat{E}_{-1}^+$ is a one-dimensional projection corresponding to (70).

Definition 5. Let

$$u_{-1}^{+}(\vec{x}) = \psi_{-1}^{+}(\vec{x})\, \mathrm{e}^{\mathrm{i}\langle \vec{P}_j(\vec{t}),\vec{x}\rangle}, \tag{77}$$

$$u_{0}^{+}(\vec{x}) = \psi_{0}^{+}(\vec{x})\, \mathrm{e}^{\mathrm{i}\langle \vec{P}_j(\vec{t}),\vec{x}\rangle}, \tag{78}$$

where ψ_{-1}^{+} and ψ_0^{+} are the periodic parts of u_{-1}^{+} and u_0^{+}, written in terms of $\hat{E}^{+}(\vec{t})$ and $E^{q,+}(\vec{t})$ as follows. First,

$$\psi_{-1}^{+}(\vec{x}) = A\left(\hat{E}^{+}\left(\vec{t}\right)_{j,j} + \hat{E}^{+}\left(\vec{t}\right)_{j-q,j} \mathrm{e}^{-\mathrm{i}\langle \vec{P}_q(0),\vec{x}\rangle}\right) \tag{79}$$

The sum in (79) has only two terms, because $\hat{E}^{+}_{lm} = 0$ if $l,m \neq 0,-q$. Note that $u_{-1}^{+}(\vec{x})$ is a linear combination of $\mathrm{e}^{\mathrm{i}\langle \vec{P}_j(\vec{t}),\vec{x}\rangle}$ and $\mathrm{e}^{\mathrm{i}\langle \vec{P}_{j-q}(\vec{t}),\vec{x}\rangle}$.

Second,

$$\psi_0^{+}(\vec{x}) = A\sum_{b\in\mathbb{Z}^n} \hat{E}^{q,+}\left(\vec{t}\right)_{j-b,j} \mathrm{e}^{\mathrm{i}\langle \vec{P}_b(0),\vec{x}\rangle}, \tag{80}$$

see (49) and definition 4.

Next, by analogy to (77) and (79),

$$u_s^{+}(\vec{x}) = \psi_s^{+}(\vec{x})\, \mathrm{e}^{\mathrm{i}\langle \vec{P}_j(\vec{t}),\vec{x}\rangle}, \tag{81}$$

ψ_s^{+} being periodic and given by the formula:

$$\psi_s^{+}(\vec{x}) = A\sum_{b\in\mathbb{Z}^n} \hat{E}_s^{+}\left(\vec{t}\right)_{j-b,j} \mathrm{e}^{\mathrm{i}\langle \vec{P}_b(0),\vec{x}\rangle}. \tag{82}$$

Definition 6. Let T be an operator in $l_2(\mathbb{Z}^n)$. Then $\|T\|_0$ is defined by

$$\|T\|_0 = \frac{1}{2}\max_r \sum_p \left(|T_{pr}| + |T_{rp}|\right). \tag{83}$$

It is obvious that $\|\mathcal{A}\mathcal{B}\|_0 \leqslant 2\|\mathcal{A}\|_0\|\mathcal{B}\|_0$.

Remind that $2\gamma_1 = 2l - n - 8\delta$, see (52).

Lemma 16. *Let $\gamma_2 > 0$, $0 < 8\delta < 2l - n$. The following estimates hold for every $m = 1, 2, \ldots$, and $\forall A \in \mathbb{C} : |\sigma||A|^2 < k^{-\gamma_2}$ when $\vec{t} \in \chi_q(k,\delta)$:*

$$\|W_m^{+} - W_{m-1}^{+}\|_* < \left(\hat{c}k^{-\gamma}\right)^m, \tag{84}$$

$$\|W_m^{+} - V\|_* < \sum_{r=1}^{m} \left(\hat{c}k^{-\gamma}\right)^r, \tag{85}$$

$$\|\hat{E}_0^{+}\left(\vec{t}\right) - \hat{E}_{-1}^{+}\left(\vec{t}\right)\|_0 < \hat{c}k^{-\gamma}, \tag{86}$$

$$\|\hat{E}_{m-1}^{+}\left(\vec{t}\right) - \hat{E}_{m-2}^{+}\left(\vec{t}\right)\|_0 < \hat{c}\left(\hat{c}k^{-\gamma}\right)^{m-1}, m \geqslant 2, \tag{87}$$

where

$$\gamma = \min\{\gamma_1, \gamma_2\}, \tag{88}$$

$\hat{c} = \hat{c}(V)$ and k is sufficiently large: $k > k_0(V, \gamma_2, \delta)$.

Corollary 17. *The sequence* $\{W_m^+\}_{m=0}^{\infty}$ *converges to a continuous and periodic potential* W^+ *with respect to the norm* $\|\cdot\|_*$. *The following estimate holds:*

$$\|W^+ - W_m^+\|_* < 2\left(\hat{c}k^{-\gamma}\right)^{m+1}. \tag{89}$$

Proof. We use an induction. For the first step $m=1$, we need to show (86) and

$$\|W_1^+ - W_0^+\|_* \leqslant \hat{c}k^{-\gamma}. \tag{90}$$

We start with (86). This is a perturbative formula for a linear operator. It is given by (56) for $m=0$ up to the notations in definition 4. To prove (90) we consider two functions $u_{-1}^+(\vec{x})$ and $u_0^+(\vec{x})$, see definition 5. Next, by (74)–(76),

$$\begin{aligned}
\|W_1^+ - W_0^+\|_* &= \left\| |\sigma| |u_0^+|^2 \right\|_* \\
&\leqslant |\sigma| \left(\| |u_0^+|^2 - |u_{-1}^+|^2 \|_* + \|u_{-1}^+\|_*^2 \right) \\
&= |\sigma| \left(\| |\psi_0^+|^2 - |\psi_{-1}^+|^2 \|_* + \|\psi_{-1}^+\|_*^2 \right) \\
&\leqslant |\sigma| \left(\| |\psi_0^+|^2 - |\psi_{-1}^+|^2 + 2\mathrm{i}\Im\left(\bar{\psi}_{-1}^+\psi_0^+\right) \|_* + \|\psi_{-1}^+\|_*^2 \right) \\
&= |\sigma| \left(\| \left(\psi_0^+ - \psi_{-1}^+\right)\left(\bar{\psi}_0^+ + \bar{\psi}_{-1}^+\right) \|_* + \|\psi_{-1}^+\|_*^2 \right) \\
&\leqslant |\sigma| \left(\|\psi_0^+ - \psi_{-1}^+\|_* \|\bar{\psi}_0^+ + \bar{\psi}_{-1}^+\|_* + \|\psi_{-1}^+\|_*^2 \right).
\end{aligned} \tag{91}$$

By (56) for $m=0$,

$$\|\psi_0^+ - \psi_{-1}^+\|_* \leqslant |A| \|\hat{E}^{+q}\left(\vec{t}\right) - \hat{E}^+\left(\vec{t}\right)\|_0 \leqslant 2|A|k^{-\gamma_1}, \tag{92}$$

Next, we notice that $\|\bar{\psi}_{-1}^+\|_* = \|\psi_{-1}^+\|_*$ and $|\hat{E}_{jj}^+| \leqslant 1$ and $|\hat{E}_{j-q,j}^+| \leqslant 1$. This implies that $\|\psi_{-1}^+\|_* \leqslant 2|A|$ because the only two nonzero elements of $\hat{E}^+(\vec{t})$ are $\hat{E}_{jj}^+$ and $\hat{E}_{j-q,j}^+$. Hence, by (92),

$$\|\bar{\psi}_0^+ + \bar{\psi}_{-1}^+\|_* \leqslant \|\bar{\psi}_0^+ - \bar{\psi}_{-1}^+\|_* + 2\|\bar{\psi}_{-1}^+\|_* \leqslant 2|A|k^{-\gamma_1} + 4|A|.$$

Since k is chosen to be sufficiently large, we can write

$$\|\bar{\psi}_0^+ + \bar{\psi}_{-1}^+\|_* \leqslant 5|A|. \tag{93}$$

Considering the last line of (91) and using (92), (93) and the condition $|\sigma||A|^2 < k^{-\gamma_2}$, we obtain that

$$\||W_1^+ - W_0^+\|_* \leqslant |\sigma| \left(\left(2|A|k^{-\gamma_1}\right) 5|A| + 2|A|^2 \right) \leqslant \quad 10|\sigma||A|^2 k^{-\gamma_1} + 2|\sigma||A|^2 \tag{94}$$

$$\leqslant 3k^{-\gamma_2}. \tag{95}$$

for sufficiently large $k > k_0(\|V\|_*, d, \gamma_1)$. This proves (84) and (85) for $m=1$.

Now, let us assume that (84)–(87) are satisfied for all $s = 1, 2, \ldots, m-1$, $m \geqslant 2$, i.e.

$$\|W_s^+ - W_{s-1}^+\|_* \leqslant \left(\hat{c}k^{-\gamma}\right)^s, \tag{96}$$

$$\|W_s^+ - V\|_* \leqslant \sum_{r=1}^{s} \left(\hat{c}k^{-\gamma}\right)^r, \tag{97}$$

$$\|\hat{E}_{s-1}^+\left(\vec{t}\right) - \hat{E}_{s-2}^+\left(\vec{t}\right)\|_0 \leqslant \hat{c}\left(\hat{c}k^{-\gamma}\right)^{s-1}. \tag{98}$$

First, let us prove (87). Considering that $H_0 + W^+_{m-1} = \hat{H}_q + (W^+_{m-1} - P_q V P_q)$ we define:

$$\hat{B}^+_{m-1}(z) = \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} \left(W^+_{m-1} - P_q V P_q\right) \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}}. \tag{99}$$

It is quite obvious that

$$\| (I - P_q) \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} \|_0 = \| \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} (I - P_q) \|_0 \leqslant k^{-\gamma_1}; \tag{100}$$

$$\|P_q \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} \|_0 = \| \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} P_q \|_0 \leqslant \frac{1}{\sqrt{d}}; \tag{101}$$

here $z \in C_1^+$, d is the radius of C_1^+, $d = \frac{1}{10}|v_q|$. Considering two above estimates and

$$W^+_{m-1} - P_q V P_q = \hat{W}^+_{m-1} + P_q \left(W^+_{m-1} - V\right) P_q, \ \text{ here } \hat{W}^+_m = W^+_m - P_q W^+_m P_q, \tag{102}$$

we obtain:

$$\max_{z \in C_1} \|\hat{B}^+_{m-1}(z)\|_0 \leqslant \left(1 + \frac{2}{\sqrt{d}}\right) \|\hat{W}^+_{m-1}\|_0 k^{-\gamma_1} + d^{-1} \|P_q \left(W^+_{m-1} - V\right) P_q\|_0. \tag{103}$$

Hence, by induction assumption (97),

$$\max_{z \in C_1} \|\hat{B}^+_{m-1}(z)\|_0 \leqslant \beta k^{-\gamma}. \tag{104}$$

where $\beta(V) = 3(1 + \frac{2}{\sqrt{d}})\|V\| + 2\hat{c}d^{-1}$. This means that

$$\|\hat{G}^+_{m-1,r}\left(k, \vec{t}\right)\|_0 \leqslant \left(c\beta k^{-\gamma}\right)^r, \tag{105}$$

where $\hat{G}^+_{m-1,r}(k, \vec{t})$ is given by (47) with $W^+_{m-1} - P_q V P_q$ instead of $\hat{V}$ and c is an absolute constant. Clearly,

$$\|\hat{E}^+_{m-1}(\vec{t})\|_0 \leqslant |\hat{E}^+(\vec{t})_{jj}| + |\hat{E}^+(\vec{t})_{j-q,j}| + \sum_{r=1}^{\infty} \|\hat{G}^+_{m-1,r}\left(k, \vec{t}\right)\|_0.$$

Since $|\hat{E}^+(\vec{t})_{jj}|, |\hat{E}^+_{m-1}(\vec{t})| \leqslant 1$ and by (105), we get $\|\hat{E}^+_{m-1}(\vec{t})\|_0 \leqslant 3$.

Now, we note that for $m \geqslant 2$:

$$\begin{aligned}
\|\hat{E}^+_{m-1}(\vec{t}) - \hat{E}^+_{m-2}(\vec{t})\|_0 &\leqslant \sum_{r=1}^{\infty} \left\|\hat{G}^+_{m-1,r}\left(k, \vec{t}\right) - \hat{G}^+_{m-2,r}\left(k, \vec{t}\right)\right\|_0 \\
&\leqslant \frac{1}{2\pi} \sum_{r=1}^{\infty} \left\| \oint_{C_1^+} \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} \left[\hat{B}^+_{m-1}(z)^r - \hat{B}^+_{m-2}(z)^r\right] \left(\hat{H}_q(\vec{t}) - z\right)^{-\frac{1}{2}} \mathrm{d}z \right\|_0, \\
&\leqslant \left(\frac{1}{2\pi} \int_{C_1^+} \left\| \left(\hat{H}_q(\vec{t}) - z\right)^{-1/2} \right\|_0^2 \mathrm{d}s\right) \left(\sum_{r=1}^{\infty} \max_{z \in C_1^+} \left\|\hat{B}^+_{m-1}(z)^r - \hat{B}^+_{m-2}(z)^r\right\|_0\right) \\
&\leqslant c \sum_{r=1}^{\infty} \max_{z \in C_1^+} \left\|(\hat{B}^+_{m-1}(z)^r - \hat{B}^+_{m-2}(z)^r\right\|_0.
\end{aligned} \tag{106}$$

Consider $\|\hat{B}^{+}_{m-1}(z)-\hat{B}^{+}_{m-2}(z)\|_0$ for any $z\in C_1^+$. Arguing as in the proof of the estimate for $\|\hat{B}^{+}_{m-1}(z)\|_0$, see (103) and (104), we obtain:

$$\|\hat{B}^{+}_{m-1}(z)-\hat{B}^{+}_{m-2}(z)\|_0\leqslant\beta\|W^{+}_{m-1}-W^{+}_{m-2}\|_0,\ \ m\geqslant 2. \tag{107}$$

On the other hand, the right-hand side of (106), which we denote here by R for simplicity, can be estimated by using (96), (104) and (107):

$$\begin{aligned}R&\leqslant c\sum_{r=1}^{\infty}\max_{z\in C_1^+}\left[\|\hat{B}^{+}_{m-1}(z)-\hat{B}^{+}_{m-2}(z)\|_0\left(\|\hat{B}^{+}_{m-1}(z)\|_0+\|\hat{B}^{+}_{m-2}(z)\|_0\right)^{r-1}\right]\\&\leqslant c\sum_{r=1}^{\infty}\beta\|W^{+}_{m-1}-W^{+}_{m-2}\|_0\left(2\beta k^{-\gamma}\right)^{r-1}\leqslant c\beta\|W^{+}_{m-1}-W^{+}_{m-2}\|_0\\&\leqslant c\beta\left(\hat{c}k^{-\gamma}\right)^{m-1}.\end{aligned} \tag{108}$$

Thus, (87) holds. Next, we consider $\|W^{+}_{m}-W^{+}_{m-1}\|_0$. Obviously,

$$\|\psi_s^+\|_*\leqslant|A|\|\hat{E}^{+}_{s}(\vec{t})\|_0. \tag{109}$$

Using (74), we easily get

$$\|W^{+}_{m}-W^{+}_{m-1}\|_0=\left\|\sigma|u^{+}_{m-1}|^2-\sigma|u^{+}_{m-2}|^2\right\|_*\leqslant|\sigma|\|\psi^{+}_{m-1}-\psi^{+}_{m-2}\|_*\|\bar{\psi}^{+}_{m-1}+\bar{\psi}^{+}_{m-2}\|_*. \tag{110}$$

Now, formula (110) can be rewritten using (82) and (109):

$$\begin{aligned}\|W^{+}_{m}-W^{+}_{m-1}\|_0&\leqslant|\sigma||A|^2\|\hat{E}^{+}_{m-1}(\vec{t})-\hat{E}^{+}_{m-2}(\vec{t})\|_0\left(\|\hat{E}^{+}_{m-1}(\vec{t})\|_0+\|\hat{E}^{+}_{m-2}(\vec{t})\|_0\right)\\&\leqslant\left(\hat{c}k^{-\gamma}\right)^m,\ \ m\geqslant 2.\end{aligned} \tag{111}$$

Therefore, (84) and (85) hold. □

3.2. Solution to nonlinear polyharmonic equation with periodic potential for $2l>n$

In this section we show that convergence of $\{W^{+}_{m}\}_{m=0}^{\infty}$ leads to convergence of the sequence of the spectral projections $\{\hat{E}^{+}_{m}(\vec{t})\}_{m=0}^{\infty}$ to that of the operator $H_0(\vec{t})+W^+$ (in the norm $\|\cdot\|_0$). The sequence of the corresponding eigenvalues $\{\hat{\lambda}^{+}_{m}(\vec{t})\}_{m=0}^{\infty}$ converges to the corresponding eigenvalue of $H_0(\vec{t})+W^+$.

Lemma 18. *Let $\gamma_2>0$, $0<9\delta<2l-n$. Suppose that $\vec{t}$ belong to the $(k^{-n+1-7\delta})$-neighborhood in K of the set $\chi_q(k,\delta)$. Then, for every sufficiently large $k>k_0(V,\gamma_2,\delta)$ and every $A\in\mathbb{C}:|\sigma||A|^2<k^{-\gamma_2}$, the sequence $\{\hat{E}^{+}_{m}(\vec{t})\}_{m=0}^{\infty}$ converges with respect to the norm $\|\cdot\|_0$ to a one-dimensional spectral projection $\hat{E}^{+}_{W}(\vec{t})$ of $H_0(\vec{t})+W^+$,*

$$\|\hat{E}^{+}_{m}(\vec{t})-\hat{E}^{+}_{W}(\vec{t})\|_0\leqslant\hat{c}\left(\hat{c}k^{-\gamma}\right)^{m+1},\ m=0,1,\ldots \tag{112}$$

The projection $\hat{E}^{+}_{W}(\vec{t})$ is given as

$$\hat{E}^{+}_{W}(\vec{t})=\hat{E}^{+}(\vec{t})+\sum_{r=1}^{\infty}\hat{G}^{+}_{W,r}(k,\vec{t}), \tag{113}$$

where. $\hat{G}^{+}_{W,r}(k,\vec{t})$ *is given by (47) with with* $W^{+} - P_q V P_q$ *instead of* $\hat{V}$. *The following estimate holds:*

$$\|\hat{G}^{+}_{W,r}\left(k,\vec{t}\right)\|_0 \leqslant \hat{c}\left(\hat{c}k^{-\gamma}\right)^{r}. \tag{114}$$

Proof. Let $B^{+}_{W}(z)$ be defined by (99) with W^{+} instead of W^{+}_{m-1}. Next, we estimate

$$\|\hat{B}^{+}_{m}(z) - \hat{B}^{+}_{W}(z)\|_0 = \|\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-\frac{1}{2}}\left(W^{+}_{m} - W^{+}\right)\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-\frac{1}{2}}\|_0.$$

By (100) and (101)

$$\|\hat{B}^{+}_{m}(z) - \hat{B}^{+}_{W}(z)\|_0 \leqslant \beta\|W^{+}_{m} - W^{+}\|_{*}. \tag{115}$$

Using corollary 17, we get

$$\|\hat{B}^{+}_{m}(z) - \hat{B}^{+}_{W}(z)\|_0 \leqslant \hat{c}\left(\hat{c}k^{-\gamma}\right)^{m+1}. \tag{116}$$

By (104),

$$\max_{z\in C_1}\|\hat{B}^{+}_{W}(z)\|_0 \leqslant \beta k^{-\gamma}. \tag{117}$$

Next, we prove (112). Let us write

$$\begin{aligned}\left\|\hat{E}^{+}_{m}\left(\vec{t}\right) - \hat{E}^{+}_{W}\left(\vec{t}\right)\right\|_0 &\leqslant \sum_{r=1}^{\infty}\|\hat{G}^{+}_{m,r}\left(k,\vec{t}\right) - \hat{G}^{+}_{W,r}\left(k,\vec{t}\right)\|_0 \\ &\leqslant c\sum_{r=1}^{\infty}\max_{z\in C_1^{+}}\left\|\hat{B}^{+}_{m}(z)^{r} - \hat{B}^{+}_{W}(z)^{r}\right\|_0 \\ &\leqslant c\sum_{r=1}^{\infty}\max_{z\in C_1^{+}}\left[\|\hat{B}^{+}_{m}(z) - B^{+}_{W}(z)\|_0\left(\|\hat{B}^{+}_{m}(z)\|_0 + \|B^{+}_{W}(z)\|_0\right)^{r-1}\right].\end{aligned} \tag{118}$$

By (104) and (115),

$$\begin{aligned}\|\hat{E}^{+}_{m}\left(\vec{t}\right) - \hat{E}^{+}_{W}\left(\vec{t}\right)\|_0 &\leqslant \sum_{r=1}^{\infty}\left[d^{-1}\|W^{+}_{m} - W^{+}\|_{*}\left(2\beta k^{-\gamma}\right)^{r-1}\right] \\ &\leqslant \hat{c}\|W^{+}_{m} - W^{+}\|_{*}.\end{aligned}$$

Hence, by (89) we get (112). Since $\|W^{+}_{m} - W^{+}\|_{*} \to 0$ as $m \to \infty$, we have that $\hat{E}^{+}_{W}(\vec{t})$ is the limit of $\hat{E}^{+}_{m}(\vec{t})$. Formulas above also prove that $\hat{G}^{+}_{m,r}(k,\vec{t})$ converges to $\hat{G}^{+}_{W,r}(k,\vec{t})$ in the norm $\|\cdot\|_0$. Now, it remains to prove (114). Indeed,

$$\begin{aligned}\|\hat{G}^{+}_{W,r}\|_0 &= \left\|\frac{(-1)^{r+1}}{2\pi\mathrm{i}}\oint_{C_1^{+}}\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-\frac{1}{2}}\left(\hat{B}^{+}_{W}(z)\right)^{r}\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-\frac{1}{2}}\mathrm{d}z\right\|_0 \\ &\leqslant \frac{1}{2\pi}\max_{z\in C_1^{+}}\|\hat{B}^{+}_{W}(z)\|_0^{r}\int_{C_1^{+}}\|\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-1}\|_0\mathrm{d}s \leqslant \left(\hat{c}k^{-\gamma}\right)^{r}.\end{aligned}$$

□

Let u^{+}_{W}, ψ^{+}_{W} be defined by definition 2 for $W = W^{+}$.

Lemma 19. *Under the assumptions of lemma 18 and, for every sufficiently large $k > k_0(V, \gamma_2, \delta)$ and every $A \in \mathbb{C} : |\sigma||A|^2 < k^{-\gamma_2}$, the function $\psi_W^+(\vec{x})$ is the limit of the sequence $\psi_m^+(\vec{x})$ in the norm $\|\cdot\|_*$. Moreover,*

$$\|\psi_m^+ - \psi_W^+\|_* \leqslant \left(\hat{c}k^{-\gamma}\right)^{m+1}, \quad m = 0, 1, \ldots \tag{119}$$

Proof. Using (112), we get

$$\begin{aligned}\|\psi_m^+ - \psi_W^+\|_* &\leqslant |A| \|\hat{E}_m^+(\vec{t}) - \hat{E}_W^+(\vec{t})\|_0 \\ &\leqslant |A| \left(\hat{c}k^{-\gamma}\right)^{m+1}.\end{aligned} \tag{120}$$

□

Corollary 20. *The sequence u_m^+ converges to $\hat{u}_W^+$ in C(Q).*

Corollary 21. *The operator $\hat{\mathcal{M}}^+$ maps the operator W^+ into W^+. In other words, $\hat{\mathcal{M}}^+ W^+ = W^+$.*

Proof. We know that $W_m^+ \to W^+$ with respect to the norm $\|\cdot\|_*$. We also know that $W_{m+1}^+ = \hat{\mathcal{M}} W_m^+$ by the equation (76). It follows that $\hat{\mathcal{M}}^+ W_m^+ \to W^+$ as $m \to \infty$. On the other hand,

$$\|\hat{\mathcal{M}}^+ W_m^+ - \hat{\mathcal{M}}^+ W^+\|_* \leqslant |\sigma| \|\psi_m^+ - \psi^+\|_* \|\bar{\psi}_m^+ + \bar{\psi}^+\|_*.$$

This implies, by (119), that $\hat{\mathcal{M}}^+ W_m^+ \to \mathcal{M}^+ W^+$ in the norm $\|\cdot\|_*$. Therefore, $\hat{\mathcal{M}}^+ W^+ = W^+$. □

Remark 22. Note that W^+ depends on $V, \sigma, A, \vec{t}$.

We use the notations $\hat{\lambda}_m^+(\vec{t})$ and $\hat{\lambda}_W^+(\vec{t})$ for the eigenvalues corresponding to $\hat{E}_m^+(\vec{t})$ and $\hat{E}_W^+(\vec{t})$, respectively.

Lemma 23. *Under assumptions of lemma 18 and, for every sufficiently large $k > k_0(V, \gamma_2, \delta)$ and every $A \in \mathbb{C} : |\sigma||A|^2 < k^{-\gamma_2}$, the sequence $\hat{\lambda}_m^+(\vec{t})$ converges to $\hat{\lambda}_W^+(\vec{t})$. The limit $\hat{\lambda}_W^+(\vec{t})$ is defined by*

$$\hat{\lambda}_W^+(\vec{t}) = \hat{\lambda}^+(\vec{t}) + \sum_{r=1}^{\infty} \hat{g}_{W,r}^+(k, \vec{t}). \tag{121}$$

Moreover,

$$|\hat{g}_{W,r}^+(k, \vec{t})| < \hat{c}\left(\hat{c}k^{-\gamma}\right)^r, \tag{122}$$

where $g_{W,r}^+$ is given by (46) with $W^+ - P_q V P_q$ instead of $\hat{V}$:

Proof. First, we prove (122). For simplicity, we denote here $\hat{E}^+(\vec{t})$ by E_0 and $E_1 = I - E_0$. Note that, for any r,

$$\oint_{C_1^+} \left(E_1 \hat{B}_m^+(z) E_1\right)^r \mathrm{d}z = 0,$$

since $E_1\hat{B}_m^+(z)E_1$ is holomorphic inside C_1^+. Consider

$$\mathcal{F}_{m,r} = \oint_{C_1^+} \left(\hat{B}_m^+(z)\right)^r \mathrm{d}z.$$

Then,

$$\begin{aligned}\mathcal{F}_{m,r} &= \oint_{C_1^+} \left[(\hat{B}_m^+(z))^r - (E_1\hat{B}_m^+(z)E_1)^r\right](\mathrm{d}z \\ &= \oint_{C_1^+} \left[\left((E_0+E_1)\hat{B}_m^+(z)(E_0+E_1)\right)^r - \left(E_1\hat{B}_m^+(z)E_1\right)^r\right]\mathrm{d}z \\ &= \sum \oint_{C_1^+} E_{i_1}\hat{B}_m^+(z)E_{i_2}\dots E_{i_r}\hat{B}_m^+(z)E_{i_{r+1}}\mathrm{d}z\end{aligned}$$

where the sum is taken over $i_1,\dots,i_{r+1} = 0,1$, and $\exists s$ such that $E_{i_s} = E_0$. Let $E_{i_{r_o}}$ be the first one to be equal to E_0. Since E_0 belongs to the trace class, it is obvious that $E_{i_1}\hat{B}_m^+(z)E_{i_2}\dots E_{i_r}\hat{B}_m^+(z)E_{i_{r+1}}$ is also in the trace class. Hence, we get by (104)

$$\|E_{i_1}\hat{B}_m^+(z)E_{i_2}\dots E_{i_r}\hat{B}_m^+(z)E_{i_{r+1}}\|_1 \leqslant \|\hat{B}_m^+(z)\|_0^{r_0}\|\hat{B}_m^+(z)\|_0^{r-r_0} \leqslant \left(\beta k^{-\gamma}\right)^r. \tag{123}$$

Thus,

$$\begin{aligned}\|\mathcal{F}_{m,r}\|_1 &= \left\|\sum \oint_{C_1^+} E_{i_1}\hat{B}_m^+(z)E_{i_2}\dots E_{i_r}\hat{B}_m^+(z)E_{i_{r+1}}\mathrm{d}z\right\|_1 \\ &\leqslant \max_{z\in C_1^+}\|E_{i_1}\hat{B}_m^+(z)E_{i_2}\dots E_{i_r}\hat{B}_m^+(z)E_{i_{r+1}}\|_1 \int_{C_1^+}\mathrm{d}s \leqslant 2\pi\left(\beta k^{-\gamma}\right)^r\end{aligned} \tag{124}$$

and $\mathcal{F}_{m,r}$ converges to

$$\mathcal{F}_r = \oint_{C_1^+} B_W^+(z)^r\mathrm{d}z \tag{125}$$

in the trace class. Let $\hat{g}_{m,r}^+$ be given by (46) with $W_m^+ - P_qVP_q$ instead of $\hat{V}$. Since $\hat{g}_{m,r}(k,\vec{t}) = \frac{(-1)^r}{2\pi\,\mathrm{i}r}\mathrm{Tr}\,\mathcal{F}_{m,r}$, it follows that $\hat{g}_{m,r}(k,\vec{t})$ converges to $\hat{g}_{W,r}(k,\vec{t})$. Then,

$$|\hat{g}_{W,r}(k,\vec{t})| = \left|\frac{(-1)^r}{2\pi\,\mathrm{i}r}Tr\mathcal{F}_r\right| \leqslant \frac{r^{-1}}{2\pi}\cdot 2\pi\left(\beta\|V\|_* k^{-\gamma}\right)^r \leqslant \hat{c}\left(\hat{c}\|V\|_* k^{-\gamma}\right)^r. \tag{126}$$

This proves (122). Now, it is clear that $\lambda_W^+(\vec{t})$ is the limit of $\{\hat{\lambda}_m^+(\vec{t})\}_{m=0}^\infty$ since

$$\left|\hat{\lambda}_m^+(\vec{t}) - \hat{\lambda}_W^+(\vec{t})\right| \leqslant \sum_{r=1}^\infty \left|\hat{g}_{m,r}(k,\vec{t}) - g_{W,r}(k,\vec{t})\right|.$$

□

The following theorem is the main result of the paper for the case $2l > n$.

Theorem 24. *Let $\gamma_2 > 0$, $0 < 9\delta < 2l - n$. Then, for each sufficiently large k: $k > k_0(V, \gamma_2, \delta)$, the following holds. If $\vec{t}$ belongs to the $(k^{-n+1-7\delta})$-neighborhood in K of the resonant set $\chi_q(k, \delta)$ and $A \in \mathbb{C} : |\sigma||A|^2 < k^{-\gamma_2}$, then, there is a pair of functions $u^{\pm}(\vec{x}, \vec{t})$ and the corresponding real values $\lambda^{\pm}(\vec{t})$, solving the equation (2) with the boundary conditions (3). Moreover, the following is true as $k \to \infty$:*

$$u^{\pm}\left(\vec{x},\vec{t}\right) = A\mathrm{e}^{\mathrm{i}\langle\vec{k},\vec{x}\rangle}\left(\psi_{-1}^{\pm}\left(\vec{x},\vec{t}\right) + \phi^{\pm}\left(\vec{x},\vec{t}\right)\right), \tag{127}$$

$$\lambda^{\pm}\left(\vec{t}\right) = \hat{\lambda}^{\pm}\left(\vec{t}\right) + O\left(k^{-\gamma}\right), \tag{128}$$

where $\psi_{-1}^{\pm}$ is given by (79) and definition 1 on page 9 (see also (70), (71)) and $\phi(\vec{x}, \vec{t})$ is periodic in $\vec{x}$, and satisfies:

$$\|\phi\|_* \leqslant \hat{c}k^{-\gamma}, \tag{129}$$

$\gamma = \min\{\gamma_1, \gamma_2\}$, $2\gamma_1 = 2l - n - 8\delta$.

From now on we set

$$u^{\pm}\left(\vec{x},\vec{t}\right) = u_W^{\pm}\left(\vec{x},\vec{t}\right), \quad \lambda^{\pm}\left(\vec{t}\right) = \hat{\lambda}_W^{\pm}\left(\vec{t}\right), \quad E^{\pm} = \hat{E}_W^{\pm}. \tag{130}$$

Remark 25. Formulas (127), (129) and (79) show that each $u^{\pm}(\vec{x}, \vec{t})$ is close to a combination of two plane waves $\mathrm{e}^{\mathrm{i}\langle\vec{k},\vec{x}\rangle}$ and $\mathrm{e}^{\mathrm{i}\langle\vec{k}-\vec{P}_q(0),\vec{x}\rangle}$ as $k \to \infty$.

Proof. We note that the function $u_W^{\pm}$ defined by (72) and the value $\lambda_W^{\pm}(\vec{t})$ given in lemma 23 by formula (121) solve the equation:

$$\left(H_0\left(\vec{t}\right) + W^{\pm}\right)u_W^{\pm}\left(\vec{x},\vec{t}\right) = \hat{\lambda}_W^{\pm}\left(\vec{t}\right)u_W^{\pm}\left(\vec{x},\vec{t}\right), \ \vec{x} \in Q, \ \ W^{\pm} = W^{\pm}\left(\vec{x},\vec{t},A\right), \tag{131}$$

and satisfying the boundary conditions (3). Using corollary 21, we can rewrite equation (131) as (2). Considering (130) we finish the proof. □

3.3. The differentiability of the eigenvalue and its spectral projection

Lemma 26. *Under conditions of lemma 16*

$$\|\nabla_{\vec{t}}\left(W_m^{\pm} - W_{m-1}^{\pm}\right)\|_* \leqslant k^{2l-1}\left(\hat{c}k^{-\gamma}\right)^m, \tag{132}$$

$$\|\nabla_{\vec{t}}\left(W_m^{\pm} - V\right)\|_* \leqslant k^{2l-1}\sum_{r=1}^{m}\left(\hat{c}k^{-\gamma}\right)^r, \tag{133}$$

$$\|\nabla_{\vec{t}}\left(\hat{E}_0^{\pm}\left(\vec{t}\right) - \hat{E}_{-1}^{\pm}\left(\vec{t}\right)\right)\|_0 \leqslant \hat{c}k^{n-1+7\delta-\gamma}, \tag{134}$$

$$\|\nabla_{\vec{t}}\left(\hat{E}_{m-1}^{\pm}\left(\vec{t}\right) - \hat{E}_{m-2}^{\pm}\left(\vec{t}\right)\right)\|_0 \leqslant k^{2l-1}\left(\hat{c}k^{-\gamma}\right)^{m-1}, \ m \geqslant 2. \tag{135}$$

Proof. The inequality (134) follows from the linear case inequality (56). To obtain the estimates (132), (133) and (135), we use the obvious inequality:

$$\left\|\nabla_{\vec{t}}\left(\hat{H}_q\left(\vec{t}\right) - z\right)^{-1/2}\right\|_0 < ck^{2l-1}d^{-3/2}, \quad z \in C_1^{\pm},$$

d being given by (43). Further considerations are analogous to that in lemma 16. □

Lemma 27. *Under conditions of lemma 18 the sequences* $\{\nabla_{\vec{t}}\hat{E}^{\pm}_m(\vec{t})\}_{m=0}^{\infty}$ *converge to* $\nabla_{\vec{t}}E^{\pm}(\vec{t})$ *in* $\|\cdot\|_0$ *and*

$$\left\|\nabla_{\vec{t}}\left(\hat{E}^{\pm}_m(\vec{t})-E^{\pm}(\vec{t})\right)\right\|_0\leqslant k^{2l-1}\left(\hat{c}k^{-\gamma}\right)^{m+1}. \tag{136}$$

The operator $\nabla_{\vec{t}}E^{\pm}(\vec{t})$ *is given as*

$$\nabla_{\vec{t}}E^{\pm}(\vec{t})=\nabla_{\vec{t}}\hat{E}^{\pm}(\vec{t})+\sum_{r=1}^{\infty}\nabla_{\vec{t}}\hat{G}^{\pm}_{W,r}(k,\vec{t}), \tag{137}$$

where

$$\|\nabla_{\vec{t}}\hat{G}^{\pm}_{W,r}(k,\vec{t})\|_0\leqslant k^{2l-1}\left(\hat{c}k^{-\gamma}\right)^{r}. \tag{138}$$

The sequences $\nabla_{\vec{t}}\hat{\lambda}^{\pm}_m(\vec{t})$ *converge to* $\nabla_{\vec{t}}\lambda^{\pm}(\vec{t})$*. The limits* $\nabla_{\vec{t}}\lambda^{\pm}(\vec{t})$ *are defined by*

$$\nabla_{\vec{t}}\lambda^{\pm}(\vec{t})=\nabla_{\vec{t}}\hat{\lambda}^{\pm}(\vec{t})+\sum_{r=1}^{\infty}\nabla_{\vec{t}}\hat{g}^{\pm}_{W,r}(k,\vec{t}), \tag{139}$$

where

$$|\nabla_{\vec{t}}\hat{g}^{\pm}_{W,r}(k,\vec{t})|<k^{2l-1}\left(\hat{c}k^{-\gamma}\right)^{r}. \tag{140}$$

Proof. The proof is analogous to those of lemmas 18 and 23 up to differentiation, lemma 26 being used. □

The next theorem easily follows from the previous lemma.

Theorem 28. *Under conditions of theorem 24:*

$$\|\nabla_{\vec{t}}\left(E^{\pm}(\vec{t})-\hat{E}^{\pm}(\vec{t})\right)\|_0<C(V)k^{2l-1-\gamma}, \tag{141}$$

$$\left|\nabla_{\vec{t}}\left(\lambda^{\pm}(\vec{t})-\hat{\lambda}^{\pm}(\vec{t})\right)\right|<C(V)k^{2l-1-\gamma}. \tag{142}$$

Corollary 29.

$$\left|\nabla_{\vec{t}}\lambda^{\pm}(\vec{t})\right|=2lk^{2l-1}\left(1+O\left(k^{-\gamma}\right)+O\left(k^{-1}\right)\right). \tag{143}$$

The corollary follows from the theorem and the obvious relation $\nabla\hat{\lambda}^{\pm}(\vec{t})=2lk^{2l-1}(1+O(k^{-1}))$.

Let us consider the surface (15) for a fixed λ_0, $\lambda_0>k_0^{2l}(V,\gamma_2,\delta)$. Note that the parts $\hat{\lambda}^{+}(\vec{t})=\lambda_0$ and $\hat{\lambda}^{-}(\vec{t})=\lambda_0$ do not intersect, since $v_q\neq 0$. Thus the deviation of the surface $\hat{\lambda}^{\pm}(\vec{t})=\lambda_0$ from the unperturbed one (V = 0) is essential. The next theorem follows.

Theorem 30. *If* $\lambda_0>k_0^{2l}(V,\gamma_2,\delta)$*, then the surface* $\lambda^{\pm}(\vec{t})=\lambda_0$ *is in the* $C(V)\lambda_0^{-\hat{\gamma}}$*-neighborhood of* $\hat{\lambda}^{\pm}(\vec{t})=\lambda_0$ *for every sufficiently large* λ_0*, here* $\hat{\gamma}=(2l-1+\gamma)(2l)^{-1}$*.*

Corollary 31. *The surfaces* $\lambda^{+}(\vec{t})=\lambda_0$ *and* $\lambda^{-}(\vec{t})=\lambda_0$ *do not intersect and located at the distance greater than* $|v_q|+O\left(\lambda_0^{-\hat{\gamma}}\right)$ *from each other as* $\lambda_0\to\infty$*.*

4. Solutions of nonlinear Scrödinger equation in dimension two

In this section, we present resonant solutions of (2) and (3) for $n = 2$, $l = 1$. The equation is

$$-\Delta u + Vu + \sigma |u|^2 u = \lambda u. \tag{144}$$

The proof of the result is analogous to that for $2l > n$. Indeed, let $S(k,\epsilon)$ be given by (16) up to replacing of $4k^{-n+2-\delta}$ by ϵ:

$$S_q(k,\epsilon) = \left\{\vec{x} \in S(k) : \left| |\vec{x}|^2 - |\vec{x} - \vec{P}_q(0)|^2 \right| < \epsilon \right\}. \tag{145}$$

We first state the geometric lemma.

Lemma 32. *Let $0 < \epsilon < \varepsilon_0$. Then, for sufficiently largek, $k > k_1(q,\varepsilon_0)$, there exists a resonant set $\chi_q(k,\epsilon) \subset \mathcal{K}S_q(k,\epsilon)$ such that, for any $\vec{t} \in \chi_q(k,\epsilon)$, the followings hold:*

$$1.\ \textit{There exists a unique} j \in \mathbb{Z}^n \textit{such that} |\vec{P}_j(\vec{t})| = k, \tag{146}$$

$$2.\ |p_j^2(\vec{t}) - p_{j-q}^2(\vec{t})| < \epsilon, \tag{147}$$

$$3.\ \min_{m \neq j, j-q} |p_j^2(\vec{t}) - p_m^2(\vec{t})| > 2\epsilon^6. \tag{148}$$

Moreover, for any $\vec{t}$ in the $(k^{-1}\epsilon^7)$-neighborhood of $\chi_q(k,\epsilon)$ in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that $|p_j^2(\vec{t}) - k^2| < 5\epsilon^7$ and the second and third conditions above are satisfied.

The set $\chi_q(k,\epsilon)$ has an asymptotically full measure on $\mathcal{K}S_q(k,\epsilon)$ as $\epsilon \to 0$, that is

$$\frac{s(\mathcal{K}S_q(k,\epsilon) \setminus \chi_q(k,\epsilon))}{s(\mathcal{K}S_q(k,\epsilon))} < c\epsilon, \quad c \neq c(k). \tag{149}$$

For all of the followings we assume that

$$\|V\|_* < \epsilon^9, \quad |v_q| > \epsilon^{10}, \quad |\sigma||A^2| < \epsilon^{11}. \tag{150}$$

Corollary 33. *If $\vec{t}$ belongs to the $(\frac{1}{8}k^{-1}\varepsilon^{10})$- neighborhood of $\chi_q(k,\epsilon)$, then for all z on the circle*

$$C_1^+ = \left\{z : |z - \hat{\lambda}^+(\vec{t})| = d\right\}, \quad d = \frac{1}{3}\epsilon^{10},$$

both of the following inequalities are true:

$$2|p_m^2(\vec{t}) - z| \geqslant \epsilon^6,\ m \neq j, j-q, \tag{151}$$

$$|\hat{\lambda}^{\pm}(\vec{t}) - z| \geqslant \frac{1}{12}\epsilon^{10}, \tag{152}$$

$\hat{\lambda}^{\pm}$ being the eigenvalues of (13).

Now, consider the map $\hat{\mathcal{M}}$ defined by (74). The following lemma can be proved by analogy with lemma 16.

Lemma 34. *There is* ϵ_0, $0<\epsilon_0\neq\epsilon_0(\lambda)$, *such that for any* $0<\epsilon<\epsilon_0$ *under the conditions (150) and for any sufficiently large k:* $k>k_1(V,\epsilon_0)$, *the following holds. Let* $\vec{t}$ *belong to the* $(\frac{1}{8}k^{-1}\varepsilon^{10})$*-neighborhood in K of the set* $\chi_q(k,\epsilon)$. *Then, for any* $m=1,2,\ldots$:

$$\|W_m^+-W_{m-1}^+\|_*\leqslant\epsilon^{11}(c\epsilon)^{m-1},\tag{153}$$

$$\|W_m^+-V\|_*\leqslant\epsilon^{11}\sum_{r=1}^{m}(c\epsilon)^{r-1},\tag{154}$$

$$\|\hat{E}_0^+(\vec{t})-\hat{E}_{-1}^+(\vec{t})\|_0\leqslant c\epsilon,\tag{155}$$

$$\|\hat{E}_{m-1}^+(\vec{t})-\hat{E}_{m-2}^+(\vec{t})\|_0\leqslant(c\epsilon)^{m-1},\tag{156}$$

where c is an absolute constant.

Corollary 35. *The sequence* $\{W_m^+\}_{m=0}^{\infty}$ *converges to a continuous and periodic function W with respect to the norm* $\|\cdot\|_*$. *The following estimate holds:*

$$\|W^+-W_m^+\|_*\leqslant 2\epsilon^{11}(c\epsilon)^m.\tag{157}$$

Proof. The following facts can be easily checked:

$$1.\ \|(I-P_q)\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}}\|_0=\|\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}}(I-P_q)\|_0<2\epsilon^{-3},\tag{158}$$

$$2.\ \|P_q\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}}\|_0=\|\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}}P_q\|_0<c\epsilon^{-5},\tag{159}$$

P_q, $\hat{H}_q$ being defined by (39) and (40). Further we use an induction. For the first step we need to show that

$$\|\hat{E}_0^+(\vec{t})-\hat{E}_{-1}^+(\vec{t})\|_0\leqslant c\epsilon,\tag{160}$$

$$\|W_1^+-W_0^+\|_*\leqslant c\epsilon^{11}.\tag{161}$$

We start with (160). This is a perturbative formula for a linear operator. We use the series (48) and (49). Indeed, we take $\hat{B}_0(z)$ given by

$$\hat{B}_0(z)=\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}}\hat{W}_0\left(\hat{H}_q(\vec{t})-z\right)^{-\frac{1}{2}},\quad \hat{W}_0=V-P_qVP_q.\tag{162}$$

Using $\|V\|_*<\epsilon^9$ and (158) and (159), we obtain:

$$\max_{z\in C_1^+}\|\hat{B}_0(z)\|_0<c\epsilon.\tag{163}$$

Now, we consider $\hat{G}_{0,r}^+(k,\vec{t})$, see (47). Applying (163) we get:

$$\|\hat{G}_{0,r}^+(k,\vec{t})\|_0\leqslant(c\varepsilon)^r.\tag{164}$$

Hence, (160) follows. Let us estimate $\|W_1^+-W_0^+\|_*$. We use again definition 5 and (91)–(93). Applying the condition $|\sigma A|^2<\epsilon^{11}$ we obtain (161).

Now, let us assume that (153)–(156) are satisfied for all $s = 1, 2, \ldots, m-1$, i.e.

$$\|W_s^+ - W_{s-1}^+\|_* \leqslant \epsilon^{11} (c\epsilon)^{s-1} \tag{165}$$

$$\|W_s^+ - V\|_* \leqslant \epsilon^{11} \sum_{r=1}^{s} (c\epsilon)^{r-1}, \tag{166}$$

$$\|\hat{E}_{s-1}^+ (\vec{t}) - \hat{E}_{s-2}^+ (\vec{t}) \|_0 \leqslant (c\epsilon)^{s-1}. \tag{167}$$

First, let us prove (156). We define $\hat{B}_{m-1}^+(z)$ by (99). Considering as in (102) and (103), we get:

$$\max_{z \in C_1} \|\hat{B}_{m-1}^+ (z)\|_0 \leqslant \|\hat{W}_{m-1}^+\|_0 \epsilon^{-8} + d^{-1} \|P_q \left(W_{m-1}^+ - V\right) P_q\|_0 < c\epsilon. \tag{168}$$

It follows $\|\hat{E}_{m-1}^+(\vec{t})\|_0 \leqslant 3$. Now, we consider (106) and further. The analog of (107) is

$$\|\hat{B}_{m-1}^+ (z) - \hat{B}_{m-2}^+ (z) \|_0 \leqslant d^{-1} \|W_{m-1}^+ - W_{m-2}^+\|_0, \ \ m \geqslant 2. \tag{169}$$

The analog of (108) is

$$R \leqslant c \sum_{r=1}^{\infty} d^{-1} \|W_{m-1}^+ - W_{m-2}^+\|_0 (c\epsilon)^{r-1} \leqslant cd^{-1} \|W_{m-1}^+ - W_{m-2}^+\|_0 \leqslant c\epsilon (c\epsilon)^{m-2},$$

$m \geqslant 2$. Thus, (156) follows. Next, we consider $\|W_m^+ - W_{m-1}^+\|_0$. Using the first line of (111), we obtain:

$$\|W_m^+ - W_{m-1}^+\|_0 < \epsilon^{11} (c\epsilon)^{m-1}. \tag{170}$$

Therefore, (153) and (154) hold. □

The convergence of the sequences of the eigenvalues and their spectral projections follow.

Lemma 36. *There is ϵ_0, $0 < \epsilon_0 \neq \epsilon_0(\lambda)$, such that for any $0 < \epsilon < \epsilon_0$ under the conditions (150) and for any sufficiently large k, $k > k_1(V, \epsilon_0)$, the following holds. Let $\vec{t}$ belong to the $(\frac{1}{8}k^{-1}\varepsilon^{10})$-neighborhood in K of the set $\chi_q(k, \epsilon)$. Then, the sequence $\{\hat{E}_m^+(\vec{t})\}_{m=0}^{\infty}$ converges with respect to the norm $\|\cdot\|_0$ to a one-dimensional spectral projection $\hat{E}_W^+(\vec{t})$ of $H_0(\vec{t}) + W^+$ and*

$$\|\hat{E}_m^+ (\vec{t}) - \hat{E}_W^+ (\vec{t}) \|_0 < (c\epsilon)^{m+1}, \ \ m \geqslant 0, \tag{171}$$

$$\|\hat{E}_{-1}^+ (\vec{t}) - \hat{E}_W^+ (\vec{t}) \|_0 < c\epsilon. \tag{172}$$

The projection $\hat{E}_W^+(\vec{t})$ is given by (113) where $\hat{E}^+$ is the spectral projection of $\hat{H}_q(\vec{t})$ corresponding to the eigenvalue $\hat{\lambda}^+(\vec{t})$ and $\hat{G}_{W,r}^+(k, \vec{t})$ is defined by (47) with $W^+ - P_q V P_q$ instead of $\hat{V}$. Moreover, the following estimate is valid:

$$\|\hat{G}_{W,r}^+ (k, \vec{t}) \|_0 \leqslant (\hat{c}\epsilon)^r. \tag{173}$$

Proof. The proof of the lemma is analogous to that of lemma 18. □

Using definition 2, we obtain the following results analogous to lemma 19, corollaries 20 and 21.

Lemma 37. *There is ϵ_0, $0<\epsilon_0\neq\epsilon_0(\lambda)$, such that for any $0<\epsilon<\epsilon_0$ under the conditions (150) and for any sufficiently large k, $k>k_1(V,\epsilon_0)$, the following holds. Let $\vec{t}$ belong to the $(\frac{1}{8}k^{-1}\varepsilon^{10})$-neighborhood in K of the set $\chi_q(k,\epsilon)$. Then, the sequence $\hat{\lambda}_m^+(\vec{t})$ converges to $\lambda_W^+(\vec{t})$ given by (121), and for all r we have*

$$|\hat{g}_{W,r}^{+}\left(k,\vec{t}\right)|<\epsilon^{10}\left(c\epsilon\right)^{r}, \tag{174}$$

where $\hat{g}_{W,r}$ is given by (46) with $W^+-P_qVP_q$ instead of $\hat{V}$.

Corollary 38. *The sequence u_m^+ converges to $\hat{u}_W^+$ in C(Q).*

Corollary 39. *The operator $\hat{\mathcal{M}}^+$ maps the operator W^+ into W^+. In other words, $\hat{\mathcal{M}}^+W^+=W^+$.*

Next, we present a solution of non-linear Scrödinger equation in the dimension two.

Theorem 40. *There is ϵ_0, $0<\epsilon_0\neq\epsilon_0(\lambda)$, such that for any $0<\epsilon<\epsilon_0$ under conditions (150) and for any sufficiently large k, $k>k_1(V,\epsilon_0)$, the following holds. Suppose $\vec{t}$ belongs to the $(\frac{1}{8}k^{-1}\varepsilon^{10})$-neighborhood in K of the resonant set $\chi_q(k,\epsilon)$. Then, there is a pair of functions $u^{\pm}(\vec{x},\vec{t})$ and the corresponding real values $\lambda^{\pm}(\vec{t})$, satisfying the equation*

$$-\Delta u^{\pm}+V(\vec{x})\,u^{\pm}+\sigma|u^{\pm}|^{2}u^{\pm}=\lambda^{\pm}u^{\pm},\ \vec{x}\in Q, \tag{175}$$

and the quasi-periodic boundary condition (3). The followings hold:

$$u^{\pm}\left(\vec{x},\vec{t}\right)=A\mathrm{e}^{\mathrm{i}\langle k,\vec{x}\rangle}\left(\psi_{-1}^{\pm}\left(\vec{x},\vec{t}\right)+\phi^{\pm}\left(\vec{x},\vec{t}\right)\right), \tag{176}$$

$$\lambda^{\pm}\left(\vec{t}\right)=\hat{\lambda}^{\pm}\left(\vec{t}\right)+O\left(\epsilon^{11}\right), \tag{177}$$

where $\psi_{-1}^{\pm}$ is as defined by (79) and definition 1 (see also (70), (71)) and $\phi^{\pm}(\vec{x},\vec{t})$ is periodic in $\vec{x}$, and satisfies:

$$\|\phi^{\pm}\|_{*}\leqslant\epsilon. \tag{178}$$

The proof of the theorem is similar to that of theorem 24, the notation (130) being used.

Theorem 41. *Under the conditions of theorem 40 the series (113) and (121) can be differentiated with respect to $\vec{t}$ retaining their asymptotic character. Moreover, the coefficients $\hat{g}_{W,r}^{\pm}(k,\vec{t})$ and $\hat{G}_{W,r}^{\pm}(k,\vec{t})$ satisfy the following estimates in the $(\frac{1}{8}k^{-1}\varepsilon^{10})$-neighborhood in $\mathcal{C}^2$ of the set $\chi_q(k,\delta)$:*

$$|\nabla_{\vec{t}}\hat{g}_{W,r}^{\pm}\left(k,\vec{t}\right)|<\left(\hat{c}\varepsilon\right)^{r}k \tag{179}$$

$$\|\nabla_{\vec{t}}\hat{G}_{W,r}^{\pm}\left(k,\vec{t}\right)\|_{0}<\left(c\varepsilon\right)^{r}\left(k\varepsilon^{-10}\right). \tag{180}$$

Corollary 42. *The followings hold for the perturbed eigenvalue and its spectral projection:*

$$\left|\nabla_{\vec{t}}\left(\lambda^{\pm}\left(\vec{t}\right)-\hat{\lambda}^{\pm}\left(\vec{t}\right)\right)\right|<C(V)\,k\varepsilon \tag{181}$$

$$\|\nabla_{\vec{t}}\left(E^{\pm}\left(\vec{t}\right)-\hat{E}^{\pm}\right)\|_{0}<C(V)\,k\varepsilon^{-9}. \tag{182}$$

Corollary 43.

$$|\nabla_{\vec{t}}\lambda^{\pm}\left(\vec{t}\right)|=2k\left(1+O\left(\epsilon\right)\right). \tag{183}$$

Let us consider the surface $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$ for a fixed λ_0, $\lambda_0 > k_1^2(V,\epsilon_0)$. Note that the parts $\hat{\lambda}^{+}(\vec{t}) = \lambda_0$ and $\hat{\lambda}^{-}(\vec{t}) = \lambda_0$ do not intersect, since $v_q \neq 0$. Thus the deviation of the surface $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$ from the unperturbed one (V = 0) is essential. The next theorem follows.

Theorem 44. *If* $0 < \varepsilon < \epsilon_0$, $\lambda_0 > k_1^{2l}(V,\epsilon_0)$, *then the curves* $\lambda^{\pm}(\vec{t}) = \lambda_0$ *are in the* $C(V)\varepsilon^{11}\lambda_0^{-1/2}$*-neighborhood of the curves* $\hat{\lambda}^{\pm}(\vec{t}) = \lambda_0$.

Corollary 45. *The curves* $\lambda^{+}(\vec{t}) = \lambda_0$ *and* $\lambda^{-}(\vec{t}) = \lambda_0$ *do not intersect and located at the distance greater than* $|v_q| + O\left(\varepsilon^{11}\lambda_0^{-1/2}\right)$ *from each other as* $\varepsilon\lambda_0^{-1/2} \to 0$.

Data availability statement

No new data were created or analysed in this study.

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