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Changing Modes of Thought Non-Euclidean Geometry and the Liberal Arts

Thomas Sibley

"The essence of mathematics lies in its freedom." - Georg Cantor

The understanding of postulates (axioms) as "self-evident truths" was forever shattered in mathematics by the introduction and understanding of non-Euclidean geometry during the nineteenth century. In addition, non-Euclidean geometry illustrates the need to transcend the intuitive models of elementary mathematics in order to think successfully about the much more abstract concepts of modern mathematics. The story of non-Eucldiean geometry plays an important role in the history of ideas and, I think, deserves to be better known.

Radical changes in the way mathematicians thought occurred during the nineteenth century. Mathematicians pursued increasingly abstract systems, even ones which seemed to contradict previously accepted mathematics. A freedom to investigate new questions developed together with a demand for rigor which surpassed previous levels.

At the same time that mathematics experienced profound changes, many other intellectual areas had major transformations. Whether one thinks of evolution in biology, Marxian analysis in economics and politics, Impressionism in art, or Freudian thought in psychology, major reformulation of thought happened in the nineteenth century. I feel that some connections exist between these changes. In particular, I wonder if the relativism which appears in many areas has some common aspects and if, perhaps, there are common causes for such shifts.

This paper has a more modest goal, confined to mathematics. This paper will provide an overview of the history of geometry into the nineteenth century, a brief

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discussion about non-Euclidean geometry and some of its consequences in physics and philosophy as well as mathematics. I leave to the reader to judge how intellectual changes in various disciplines parallel or diverge from those I will describe in mathematics.

A Sketch of Geometry before Non-Euclidean Geometry

"As to writing another book on geometry, the middle ages would as soon have thought of composing another New Testament." - Augustus De Morgan

"We hold these truths to be self-evident..." - Thomas Jefferson

From Pythagoras (circa 500 BC) to Kant (1724-1804), mathematics was considered unquestionably true. Thus we have 2,300 years of this orientation. Euclid's synthesis of geometry (c. 300 BC) comes early on, and no one before 1800 doubted the truth of Euclid's postulates (axioms) and theorems. However, throughout this span of time, commentators focused on one "flaw" in Euclid— one postulate, the fifth one, was not self evident.

Postulate 1. To draw a straight line from any point to any point.
Postulate 2. To produce a finite straight line continuously in a straight line.
Postulate 3. To describe a circle with any center and distance.
Postulate 4. That all right angles are equal to one another.
Postulate 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Heath,195-202)

I suspect you too would not consider this fifth postulate, illustrated in Figure 1, "self-evident" by any stretch of the imagination. Euclid chose this wording to avoid the dangers of infinite lines. Aristotle, among other Greeks, realized the shakiness of reasoning about "actual infinities," as opposed to "potential infinities." Euclid's straight lines are indefinite in length, although "potentially infinite" in that there is no limit to their lengths. Thus Postulate 5 guarantees that the lines will meet in a finite, although indefinite, distance. In modern terms, this postulate is equivalent to saying that for two lines to be parallel, any transversal must cut them such that alternate interior angles add up to 180°, i.e. two right angles. (See Figure 2.)

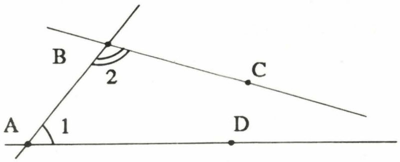

Figure 1. *If $\measuredangle 1 + \measuredangle 2$ is less than 180°, lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ will meet somewhere on the right.*

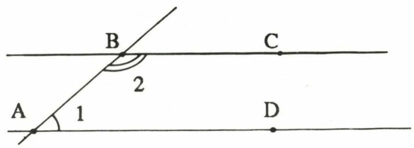

Figure 2. *If $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are parallel, then $\measuredangle 1 + \measuredangle 2 =$ 180°.*

Many mathematicians tried but failed to prove Euclid's fifth postulate as a theorem, using only his other four postulates. For the most part, they either explicitly or implicitly used equivalent postulates, such as Playfair's, which is now used in geometry books. It reads "Given a line k and a point P not on k, there is just one line m parallel to k and passing through the point P" (Kline, *Thought* 865). Before 1800, the one who came closest to realizing that the fifth postulate could not be proven from the others was an Italian mathematician named Gerolamo Saccheri (1667-1733). His approach was to start from the negation of the fifth postulate and look for a contradiction. He deduced increasingly bizarre consequences, such as the existence of two straight lines which approach each other but never cross. (See figure 3.) But he found no contradiction. Finally, he concluded, "the hypothesis... is absolutely false, because it is repugnant to the nature of the straight line" (Bonola,43). Saccheri missed being the inventor of non-Euclidean geometry because he couldn't transcend the world view asserting the truth of Euclidean geometry. That world view seemed increasingly convincing throughout the 1700s.

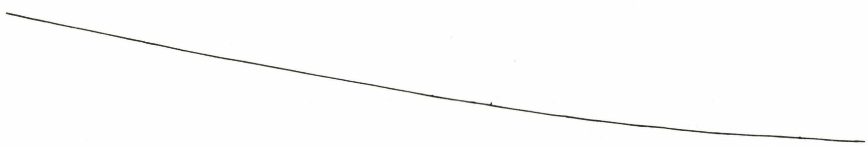

Figure 3. *Two "straight" lines which approach each other but never cross - - one of the consequences of Saccheri's investigations.*

The eighteenth century, the Age of Enlightenment, built on the assumed truth of mathematics in two ways pertinent to our discussion. First of all, Kant classified our (Euclidean) geometric notions of space as necessary prerequisites to perception. He argued that mathematical knowledge was synthetic a priori, that is gained without experience, but nevertheless providing new information about our world

(Kline, *Thought* 862). Secondly, the astounding success of Newton's calculus and physics convinced his successors that mathematics was not just true in some metaphysical sense, but also in a palpable sense. The ideal world of mathematics, it seemed, was the real world. Some felt, echoing Galileo's sentiments, that God had written the book of nature in mathematical terms. The confluence of the philosophical certainty and physical centrality of mathematics bolstered confidence in an age of reason guided by mathematics. During the eighteenth century, mathematicians eagerly extended mathematics and found numerous connections with physics. Indeed, the term "natural philosophy" then current made it less necessary to distinguish mathematics from physics (Kline, *Thought* 619-621).

The physical meaning of much of the mathematics developed in the eighteenth century was sufficiently convincing that the rigorous deductive methods of Greek geometry seemed superfluous. When mathematical equations could accurately predict complicated events which were previously inexplicable, no one felt any need to scrutinize the mathematics. It took the shock of the radically different mathematical results of the nineteenth century, like non-Euclidean geometry, to force mathematicians to reintroduce rigor (Kline, *Thought* 617).

The first person to break out of the world view of Euclidean geometry and its unquestionable truth was Carl Friedrich Gauss (1777-1855), the greatest mathematician since Newton. However, Gauss never published anything on non-Euclidean geometry because he feared ridicule, a reflection of the dominance of Kantian thinking about geometry. He did absorb the work of his predecessors, and through correspondance he passed on their work and his own insights. Nikolai Lobachevsky (1793-1856) and John Bolyai (1802-1860), the two young mathematicians who did publish works on non-Euclidean geometry as an independent geometry, were greeted with silence for a number of years after their publications in 1829 and 1832, respectively (Kline, *Thought* 869-879). I will fill in that period of silence with a sketch of their results.

A Sketch of Non-Euclidean Geometry

"The most suggestive and notable achievement of the last century is the discovery of non-Euclidean geometry." - *David Hilbert*

Hyperbolic geometry, the non-Euclidean geometry which Gauss, Lobachevsky and Bolyai created, retained the first four postulates of Euclid and changed the fifth postulate to the form given below.

Revised Postulate. Given any line k and any point P not on k, there are at least two lines through P which do not intersect k.

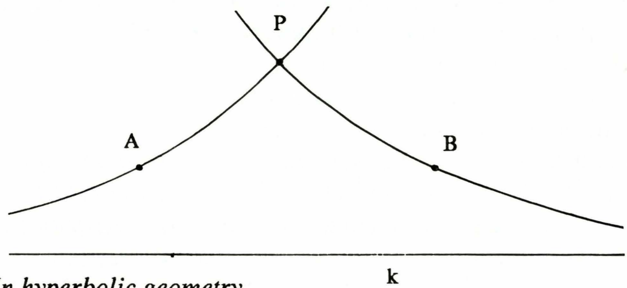

Figure 4. *In hyperbolic geometry, it is possible for both* $\overleftrightarrow{AP}$ *and* $\overleftrightarrow{BP}$ *to miss line k.*

This postulate makes the notion of "parallel lines" very different in hyperbolic geometry. In fact, one can show that through the point P in the revised postulate, there must be infinitely many lines which do not intersect k. A variety of other consequences result, including the many which Saccheri found. (See Figure 3.) The most startling consequence is the theorem that the angles of a triangle do not add up to 180°, as they do in Euclidean geometry. The theorem below goes even further, relating the sum of the angles with the area of the entire triangle. (See Figure 5.) The bigger the area of the triangle, the smaller the sum of the angles. One consequence of this is that there is a maximum area for any triangle. Since the sides of triangles can become indefinitely long, this consequence seems paradoxical.

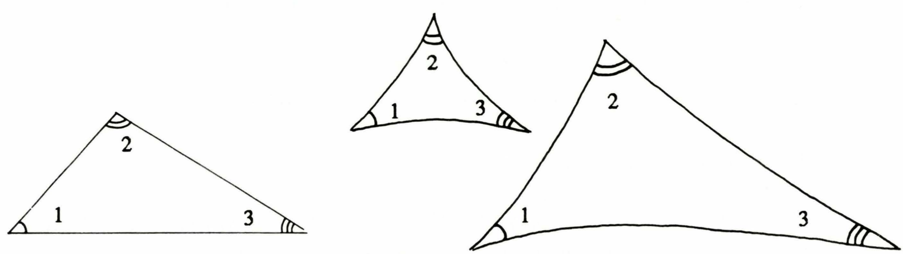

Figure 5a. *In Euclidean geometry, we always have* $\measuredangle 1 + \measuredangle 2 + \measuredangle 3 = 180°$.

Figure 5b. *In hyperbolic geometry,* $\measuredangle 1 + \measuredangle 2 + \measuredangle 3 < 180°$ *and bigger triangles have smaller sums.*

Theorem. The difference, $180° - (\measuredangle A + \measuredangle B + \measuredangle C)$, between 180° and the sum of the angles of a triangle in hyperbolic geometry is proportional to the area of the triangle.

This theorem stated something measurable about a fairly simple object which we can construct approximately in the real world. Gauss considered the angles formed by three mountains. He found that the accuracy of the measurements was not good enough to tell if the sum of those angles was greater than, less than or equal to 180°. Thus, the first test of the reality of hyperbolic geometry proved inconclusive (Losee 169 and Kline, *Loss* 85). By the end of the nineteenth century, mathematicians realized the naivete of empirically deciding which geometry was correct because one must make physical assumptions as well as test mathematical relations.

All geometry students encountering hyperbolic geometry for the first time experience perplexity with the series of theorems which fly in the face of everything they have been taught about geometry since grade school. They relive, in effect, the experience of Saccheri. One thing hampering the understanding of modern mathematics is the overly naive sense of model which students have from high school. Until we need a more sophisticated notion, we assume mathematical terms describe the intuitive world we experience. Thus a "line" in geometry "ought" to be an abstraction of taut threads or edges of boards. In such an intuitive model of geometry, postulates simply express accepted, even obvious, properties. However, the revised postulate and its consequences do not fit with the comfortable pictures we have of space, as codified in Euclidean geometry. Modern mathematics uses postulates to define abstractly what we mean by terms which are not supposed to carry strong connotations from overly naive models. Although this modern understanding of mathematics was not caused solely by the creation of non-Euclidean geometry, the changes in geometry were symptomatic of the profound change which happened to mathematics in the nineteenth century.

Mathematics after the Advent of Non-Euclidean Geometry

"But in the present century, thanks in good part to the influence of Hilbert, we have come to see that the unproved postulates with which we start are purely arbitrary. They MUST be consistent; they HAD BETTER lead to something interesting." - Julian Coolidge

The explosion of mathematical activity since 1800 admits no easy summary. However, three general aspects deserve our attention here because they represent seminal changes in nineteenth century mathematics which connect with non-Euclidean geometry. First, large parts of mathematics have become very abstract, exploring formal systems far more general and less intuitive than previous systems. Second, mathematicians have become much freer in inventing new systems, even ones which explicitly contradict more intuitive systems. (It should be noted that actual mathematical systems do not start from purely arbitrary postulates, despite the preceding quote of Coolidge. Mathematicians choose their postulates to reflect what they are investigating.) Third, the demand for rigorous proofs has pervaded much of mathematics. A need to understand what constitutes a proof in an abstract system has led to an investigation of logic itself, opening new doors to mathematics. Although these changes have happened in many areas within mathematics, the effects in geometry have been quite noticeable and widely discussed.

The first notable response to the advent of hyperbolic geometry came in 1854 in a lecture delivered by Georg Bernhard Riemann (1826-1866) for his introductory lecture at Gottingen University. Since the audience was not a mathematical one, only Riemann's teacher,the aging Gauss, caught the point of this lecture, entitled "On the Hypotheses which underlie Geometry." However, Riemann published this talk later, focusing geometrical thought on a new field, differential geometry, and initiating an active debate on non-Euclidean geometries. He realized that the work of Gauss, Lobachevsky and Bolyai was more than playing abstractly with a meaningless postulate. In essence, he saw that the revised postulate implied space had to be shaped differently than what Euclid's fifth postulate implied. He then articulated how one could create infinitely many different geometries, each with its own shape. Riemann's vision was so broad that he conceived of these geometries as having any number of dimensions, not just the one, two or three dimensions we can visualize (LeCorbeiller 128-133).

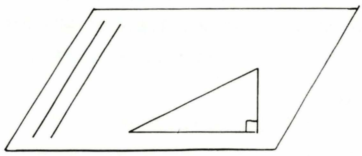

Figure 6. *A portion of the Euclidean plane with parallel lines and a right triangle.*

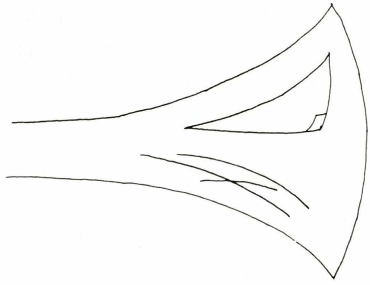

Figure 7. *A portion of the hyperbolic plane with lines illustrating the revised postulate and a right triangle.*

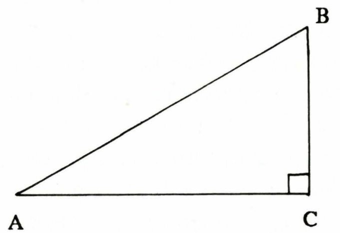

Figure 8. *We can find the hypothenuse AB in terms of the lengths of the sides AC and BC: (AB)2 = (AC)2 + (BC)2.*

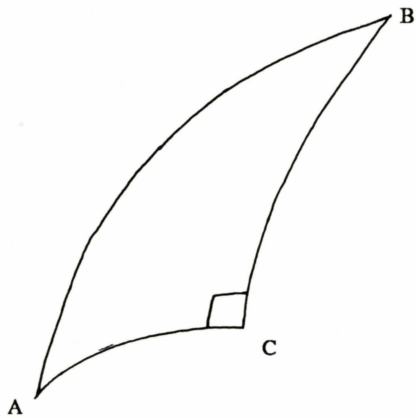

Figure 9. *The general formula is more complicated. The values of f, g and h depend on the curvature of the geometry: (AB)2 = f(AC)2 + g(AC)(BC) + h(BC)2.*

Differential geometry, the field that Riemann and Gauss before him started, investigates geometries by looking at how they behave in small regions. For example, the Euclidean plane is flat, while hyperbolic geometry has to bend like a saddle in order to accomodate all the peculiarities stemming from the revised postulate. (See Figures 6 and 7.) It would be easiest to think of the geometries Riemann considered, including hyperbolic geometry, as surfaces inside ordinary Euclidean space; but unfortunately most such geometries do not fit inside ordinary Euclidean three-dimensional space. For example, only portions of hyperbolic geometry can fit inside Euclidean three-dimensional geometry due to the bending of hyperbolic geometry. To free himself from Euclidean assumptions Riemann needed to articulate the hypotheses which underlie geometry. What made it possible for all of these things to be geometries? Riemann decided that to do geometry, he needed to be able to measure distances along different directions in the geometry and to be able to describe how these measurements interacted. In familiar Euclidean plane geometry, the interaction is described by the Pythagorean theorem, illustrated in Figure 8. More complicated geometries require more complicated interactions as in Figure 9.

Consider musical notes as an example of a geometry in this sense. (Riemann would have had no difficulty with this idea.) One dimension of measurement for a note would be the pitch. Another would be its duration. A third would be its intensity. In a more sophisticated model, one could include the intensities of the various harmonics which give each instrument its characteristic timbre. I have no idea what the interactions between these various measurements are, but in principle, music qualifies as a geometry in Riemann's sense.

Riemann's vision of geometry clearly was very abstract and inclusive. Others approached non-Euclidean geometry from different standpoints, including Felix Klein (1849-1925), who exemplified another fruitful approach to understanding the variety of geometries which appeared during the nineteenth century. He took as fundamental the possible symmetries of a geometry, using them to study the various properties of a geometry and, more importantly, to relate different geometries to one another. In Euclidean geometry, the symmetries include rotations, translations (sliding motions) and reflections, shown in Figure 10. A symmetry in general is a permissible motion of the geometry which makes one shape move onto another shape in that geometry. Hyperbolic geometry has symmetries as well, although they are harder to picture. Instead, consider the geometry of a sphere, which allows rotations and reflections, as in Figure 11.

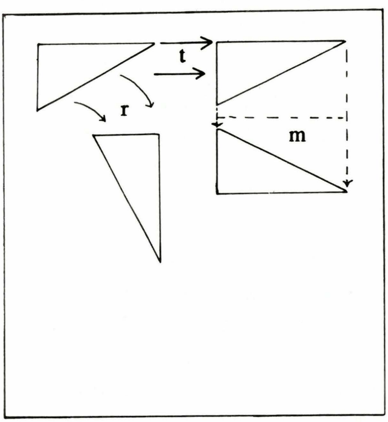

Figure 10. *In Euclidean geometry, a translation (t), a rotation (r) and a mirror reflection (m) move a triangle or other shape to a congruent shape.*

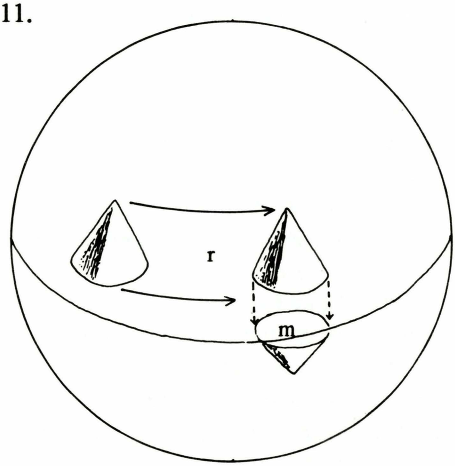

Figure 11. *In spherical geometry, a rotation (r) and a mirror reflection (m) move a triangle or other shape to a congruent shape.*

This focus on the symmetries allowed mathematicians to abstract the concept of congruence from Euclidean figures and to turn it into an idea which could apply to geometries which didn't have the concepts of distance and angle. For example, projective geometry includes symmetries which distort shapes, such as the ones artists use in drawing in perspective. The circular rim of a glass might appear as an ellipse in a drawing to convey the angle of viewing. (See Figure 12.) Such a change goes beyond Euclidean geometry, since it changes the distances of the original points of the circular rim to the distances of the points in the sketch. Klein and others revealed the common ground underlying Euclidean, hyperbolic, spherical and projective geometries.

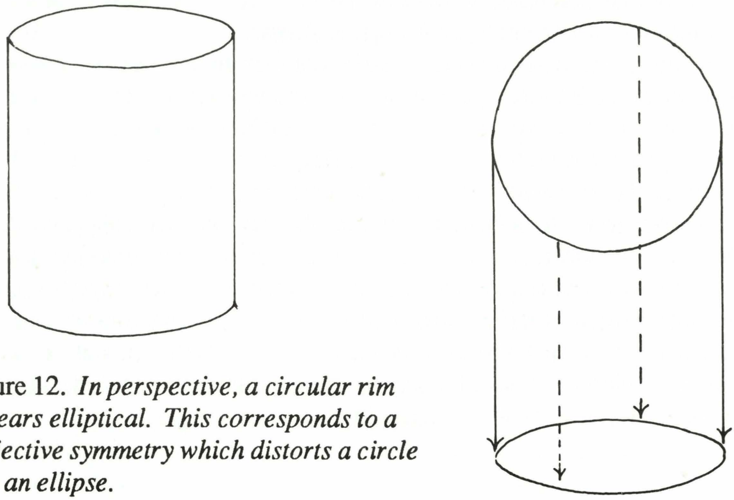

Figure 12. *In perspective, a circular rim appears elliptical. This corresponds to a projective symmetry which distorts a circle into an ellipse.*

Mathematicians in all areas built more abstract systems and used the abstractness to free their minds. In algebra, arithmetic systems appeared where a*b does not always equal b*a. In analysis, mathematicians imagined strange curves bent so badly at every spot they could not be drawn.

The increase in abstraction required a corresponding increase in the rigor of mathematical arguments. No longer could a mathematician rely on an intuitive model to reveal the essential idea behind an argument. Mathematicians made explicit the more sophisticated reasoning required for abstract mathematics. Aristotle's insights into logic were superceded by a formal logic both more powerful and flexible than his syllogisms and more in tune with the abstract nature of the mathematics.

The Influence of Mathematics

"Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true." - Bertrand Russell

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." - Albert Einstein

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?" - Albert Einstein

While mathematics incorporated abstraction, rigor and new freedom into its creations, other areas incorporated the new mathematics in varying ways. I will consider two aspects which were influenced by the advent of non-Euclidean geometry, Poincaré's ideas in philosophy of science and Einstein's theory of relativity.

In the latter part of the nineteenth century, many wrestled with the questions "Which geometry is true?" and "Which geometry represents the real world?" Henri Poincaré (1854-1912) provided a very modern resolution to this quandary, compared with Gauss' efforts to determine the answer experimentally by measuring angles formed by mountains. Poincaré propounded a philosophy of science now called conventionalism. He said that we cannot test the truth of a particular concept independently of the theory and interpretations of which it is a part. In the case of geometry, we would like to know if, in the "real world," a triangle has exactly 180° for the sum of its angles or not. Poincaré argued that in order to answer this, one must first decide what "lines" are in "reality." He pointed out that we could postulate lines to be Euclidean and then build our physics based on that, or we could pick another postulated form for the underlying geometry and build our physics on that. While either would work, the simpler system should be preferred. Poincaré thought that Euclidean geometry would always be preferred to any other geometry due to its simplicity, but none had any claim to truth. He pointed out that there was no empirical way to test the different geometries because one had to use real phenomena, not mathematical lines. For example, Gauss' attempt to measure the angles of triangles whose corners were mountains used the path of light as a straight line. But how could one measure the straightness of the light path itself? (Einstein showed how to determine that light paths are not straight, as in Figure 13.) In other words, the assumption about the lines ties in with assumptions about physical phenomena, like the path of light. Hence, the reality of Euclidean geometry is untestable outside the context of other, more empirical assumptions. This does not make the assumptions arbitrary, since science must conform with experiment.

The theory of relativity of Albert Einstein (1879 - 1955) has justifiably caught the imagination of popular Western culture and has made enormous contributions to scientific thought. However, the popular culture does not accord the same excitement to the mathematics which is the language of relativity. Einstein's special theory of relativity has as its natural model a four-dimensional geometry, Minkowski geometry, which has striking similarities to hyperbolic geometry. From Poincaré's point of view, this non-Euclidean geometry provides a simpler geometrical base for the physics. However, Einstein did not stop there. He went on to investigate how gravitational forces "bend" space. To describe these deep insights, Einstein and all who work in relativity drew on the differential geometry developed by Riemann. Indeed, the curvature of space needed to describe relativistic effects is just the thing Riemann's geometrical approach sought to describe. Hence Riemann's abstract vision of geometry has been mathematically fruitful in its own right and has flowered wonderfully in its applications.

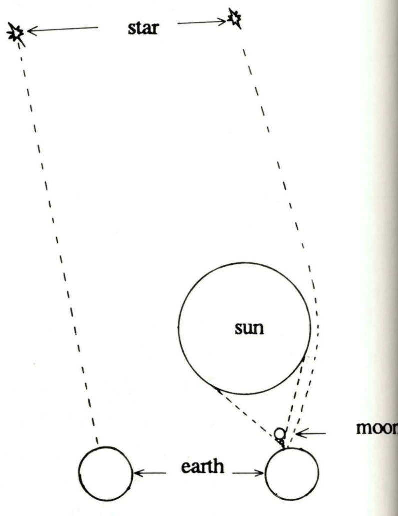

Figure 13. Sir A. Eddington verified the bending of light beams in the presence of gravity. During an eclipse of the sun in 1919, he observed a star which "should" have been hidden by the sun. Instead, the light beam was bent by the gravitation of the sun. One can model this bending by using a geometry which has a curvature so that the light beam naturally follows a curved path. The diagram at the far left shows the ordinary path of light, while the other diagram shows the sun's effect.

Riemann's geometry is but one example of the rich dividends abstraction has paid in mathematics. In every area of mathematics, there are abstract systems which have led to deep mathematical insights and have at the same time provided powerful models in other areas. Finite geometries appear in the design of statistical experiments; and the convoluted curves of analysis, renamed as fractals, find applications as models of lungs, geological fault lines and coastlines. Using the very word "model" for an application of mathematics points out a change in how we see mathematics. The entire body of mathematics does not have to be the ideal laws governing the universe. Instead, an abstract mathematical system may provide the means to recognizing an underlying connection between seemingly disparate phenomena.

Mathematics and the History of Ideas

"Genius is the willingness to test the strangest alternatives." - Anon.

Other areas of nineteenth century Western culture underwent profound change at the same time as mathematics did. Simply dropping names like Marx, Darwin and Freud evokes images of how differently we see the world compared to people of the Enlightenment. What connections and divergences are there between the change of viewpoint in mathematics and the changes in other areas? Is it possible to find underlying causes or contributing factors?

I make no pretenses of being a historian, but I think that there are important connections between the changes in various disciplines since the last half of the nineteenth century. The willingness of great minds in a variety of disciplines to take radically different approaches to understanding their subjects seems more than coincidental. Perhaps the fervor and excess of political turmoil from 1789 to 1848 transformed the benign glow of the Enlightenment into intellectual revolution. The simultaneous unfolding of the Industrial Revolution added its stimulus, as did the concentrating of intellectual thought in greatly altered universities.

In any case, I think that the twenty-five year pause between the initial publishing of non-Euclidean geometry and its active integration into the mainstream of mathematical discourse may point to more than historical peculiarities in the mathematical community. The long and active investigation of Euclid's fifth postulate should have ensured an interested, if critical, reception to hyperbolic geometry. However, I think the general mathematical climate in 1830 was not ready to handle such non-intuitive mathematics. In a similar vein, intimations of four-dimensional geometry appeared about this time, but it was in the last half of the nineteenth century that mathematicians incorporated higher-dimensional space into the realm of mathematics. Other areas of mathematics contributed similar examples of this change in acceptance of abstract, non-intuitive mathematics build on rigorous foundations.

Somewhere around 1850, it seems that European thought crossed a watershed which has opened our culture to profound changes. Beethoven may have seemed impetuous to the waning Rococco period, but Stravinsky would have been utterly frightening. The same seems to hold in art, literature and the sciences. I find the transitions in mathematics as exciting and fruitful as the changes in other disciplines. While the beautiful proofs of Gauss would have almost certainly found an appreciative audience in Newton, the abstract mathematics of today seems too far removed to be comprehended by that great mathematician. The serene and intuitive truth of Euclidean geometry has yielded to an abstract theory positing infinitely many possible geometries.

Paradox

Not truth, nor certainty. These I forswore
In my novitiate, as young men called
To holy orders must abjure the world.
"If..., then...," this only I assert;
And my successes are but pretty chains
Linking twin doubts, for it is vain to ask
If what I postulate be justified,
Or what I prove possess the stamp of fact.

Yet bridges stand, and men no longer crawl
In two dimensions. And such triumphs stem
In no small measure from the power this game
Played with the thrice-attenuated shades
Of things, has over their originals.
How frail the wand, but how profound the spell!

Clarence R. Wylie, Jr.

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For Further Reading

Richards, Joan L. *Mathematical Visions: The Pursuit of Geormetry in Victorian England.* Boston: Academic Press, 1988.
This excellently written book explores the English cultural and mathematical content and effects of the revolutionary changes in geometry in the nineteenth century.