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# Fantasy on a Baseball Seam

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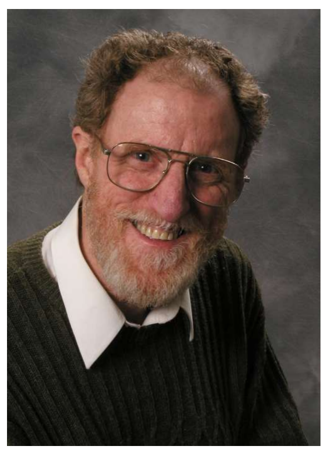

Thomas Q. Sibley (tsibley@csbsju.edu) recently retired from teaching at St. John's University and the College of St. Benedict. Prior to that he taught in the Democratic Republic of the Congo and Liberia. He received his Ph.D. from Boston University. He has authored three mathematics textbooks as well as articles, including several coauthored with his undergraduate students.

Sophia and Jamal are sitting in the student lounge when Jamal notices another student watching baseball on his laptop.

J: Sophia, have you ever thought about how baseballs are constructed?

S: What do you mean?

J: Well, when they make a baseball, two identical leather pieces are wrapped around the inside core and sewn together, creating the seam. And, it turns out that the bumpiness of the seam on the surface enables the pitcher to do cool things with the path of the ball, like throwing a curveball.


Figure 1. A baseball.

S: That is interesting, but I am more interested in the path of the stitches. I wonder if there are polyhedra whose surfaces can be split into two pieces like that.

J: Good question. Let's start by investigating a cube.

S: Well, cubes have six faces, so if we want to split a cube into two congruent pieces, each piece will need to have 3 faces. Does it matter how we split these faces up?

J: Let me think. There are only two ways that I can think of to split up the faces. One would be when three square faces meet at a common vertex and the other would be when three faces form a row if you bend them on their common edges to lay them flat. That mimics the way the leather pieces of a baseball start as flat pieces. (See Figure 2.)

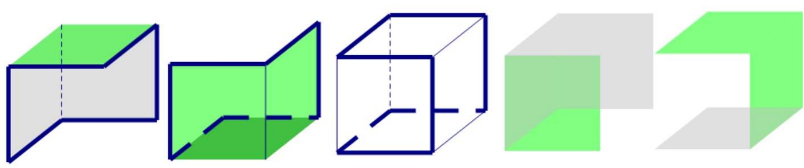

Figure 2. Two ways to split a cube.

J: I think we need to create a solid definition of what a baseball seam is. When we look at our two proposed seams, the second one, the seam for the three faces in a row curve around like a seam on the baseball. I feel like the second one is definitely a baseball seam, but I do not think the first one should be a seam because we can't unbend it along edges to fold flat.

S: I agree. We should try out some different possibilities to see what properties seem to fit our intuition.

J: I think it would be best to examine the edges when identifying our seams. Our first cube seam has six edges and the second has eight. Let's next look at a square pyramid. How about these five edges for its seam? (See Figure 3.)

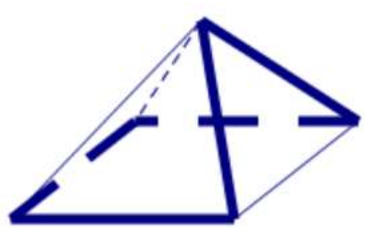

Figure 3. A square pyramid and seam.

S: As usual, you seem to have a knack for suggesting counterexamples. I really don't want that one to qualify.

J: What's wrong with the pyramid example?

S: The edges split the surface into three triangles for one piece and a triangle and a square for the other. They aren't congruent, but then they could hardly be since there is only one square.

J: Good observation. I was originally making sure that each part could unfold flat along edges, but I completely overlooked making sure they had the same type of faces. Well, the cube example certainly doesn't have that defect. How about this preliminary definition?

**Definition (Seam).** A *seam* of a polyhedron is a cycle of edges that separates the faces of the polyhedron into two connected congruent pieces.

The one problem with this definition is that the six-edge seam of the cube still satisfies it.

S: Well, with the six-edge seam, the two halves don't spread out flat without overlapping. With the second way of splitting the cube, each of the strips of three squares naturally folds flat.

J: Maybe we'll not get too technical about folding flat and just include that, for now, to see where it takes us.

**Definition (Baseball seam).** A *baseball seam* of a polyhedron is a cycle of edges that separate the faces of the polyhedron into two connected congruent pieces that can fold flat along edges.

S: I think that fits my intuition pretty well, and I'm game to avoid defining folding flat. Since a pyramid with a square base has no chance, we could try a triangular pyramid. With four triangles, we can certainly split them up evenly. I think a cycle of four edges works for me. (See Figure 4.)

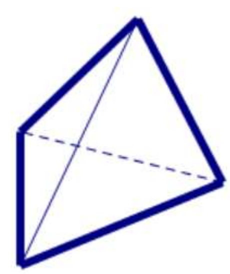

Figure 4. A seam for a triangular pyramid.

J: I agree! In general, what can we say about polyhedra with seams?

S: Even with just the first definition, they need to have an even number of each type of face. Otherwise, the two parts split by the edge can't be congruent, let alone fold flat. That rules out all pyramids except triangular ones.

J: The cube is a pretty special shape, as is a regular tetrahedron. Is there a whole family of polyhedra with seams?

S: Great question! Prisms could work since they have two congruent bases. Oh, and they would need an even number of rectangular sides. In fact, a cube is a special case of a square prism. The next size up is a hexagonal prism (as in Figure 5). Each base must join up with three of the rectangular faces. And for a baseball seam adjacent rectangles can’t be connected with the same base or the pieces won’t fold flat.

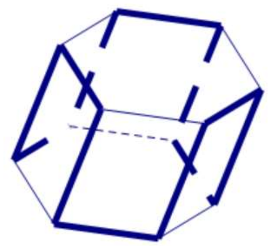

Figure 5. A seam on a hexagonal prism.

J: What a wonderful observation! Maybe we could apply this to antiprisms as well. (These are like prisms, only the bottom base is rotated from the top base so that the sides are isosceles triangles, as in Figure 6.) Looking at a hexagonal antiprism, we can construct a seam outlining one hexagonal base and the six triangular faces adjacent to it. Conveniently, the triangles attached to a base aren’t adjacent to each other so the pieces would fold flat.

S: Fascinating! And since antiprisms always have an even number of triangular faces, all antiprisms would have baseball seams!

Figure 6. A hexagonal antiprism.

J: Which polyhedra should we examine next?

S: What about bipyramids (two pyramids with their bases “glued together”)? We could start with the triangular bipyramid. (See Figure 7.) Since it has six faces, we can create a seam with three faces for each piece. How about this seam consisting of five edges?

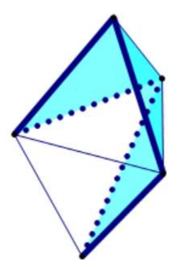

Figure 7. A bipyramid with a seam.

J: Great! We can apply that same method to a square bipyramid, or any bipyramid for that matter. One piece has $n - 1$ triangles on the top together with one on the bottom.

S: Wait, what about a rectangular (non-square) bipyramid? Not all its faces are congruent. (See Figure 8.)

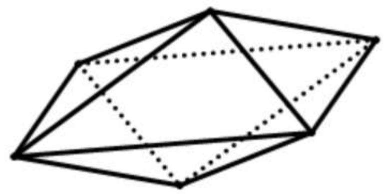

Figure 8. A rectangular bipyramid.

J: Good point. Though we should still be able to construct a seam, we just might need to do it a little differently. Instead of constructing the seam from one face on the top portion and all but one face on the bottom, we can create the seam by taking two adjacent faces from the top portion and two faces from the bottom portion shifted over so they are connected by just one edge. That way there will be one wide and one narrow triangle on the top and one of each on the bottom for each piece.

S: Nice. So far it looks like any polyhedron with an even number of each type of face has a baseball seam. How could it go wrong?

J: Let's start with something simple and familiar. How about a rectangular box with different lengths in each dimension? We can try out the baseball seam that worked for a cube.

S: You are right. Nice counterexample! We don't get congruent pieces. The way that doesn't fold flat does give congruent pieces and so fits the first definition. So, we can get the strong definition of a baseball seam to fail. How about the weaker definition of a seam?

(There is a pause as Jamal and Sophia draw and reject a variety of shapes.)

S: How about using an odd shaped hexagon (as in Figure 9) for the two bases of a prism? It has four short sides and two long ones, but they don't alternate. It is possible for us to pick two of the short rectangles and one long rectangle to go with each base. However, no matter how we pick these, the two halves can't be congruent because the spacing between the two short rectangles can't be the same for the two pieces.

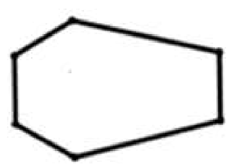

Figure 9. The base of a prism with no seam.

J: Nice counterexample. While you were dreaming up that one, I was noticing that in our previous examples that fold flat, the seam needs to go through each vertex. That way the faces around each vertex are split between the two pieces. Thus, the number of edges in a seam equals the number of vertices in the entire polyhedron.

S: Great observation! That means that baseball seams are Hamiltonian cycles. I was reading about them in West [3, 218 ff] People have been playing with them even before William Rowan Hamilton made a puzzle about them. And I see how Hamiltonian cycles fit with the pieces folding flat. I recall that Euler's formula links the total number of edges with the number of vertices and faces: $V - E + F = 2$. (See West [3, 255].) Perhaps baseball seams connect with that.

J: Let’s count the edges. There are $V$ of them on the baseball seam. We can split the faces into two equal sets, so there are $\frac{F}{2}$ faces in each piece. Also, the faces in each piece are connected, so there needs to be $\frac{F}{2}-1$ edges to make the connections. Altogether, that gives $2(\frac{F}{2}-1)=F-2$ edges within the pieces. And $V+F-2$ edges altogether. So, $V-E+F=2$. Nice! That means the two pieces and a baseball seam always account for all the edges of the polyhedron.

S: That is a beautiful connection. Will it always work? Do you have some stranger polyhedra that will test this out?

J: Let’s take a pause to hunt.

[They meet somewhat later.]

J: In my research, I found that there exist polyhedra with no Hamiltonian cycles at all. (See Coxeter [1, 8].) One modifies the triangular bipyramid of Figure 7 by erecting a pyramid on each face as in Figure 10. This gives a total of 11 vertices and 18 congruent isosceles triangles. A really beautiful proof by contradiction shows that there is no Hamiltonian cycle. If there were, it would have to go into and out of each of the six vertices of the added pyramids. But then it would need to go into and out of 6 of the original vertices of the bipyramid. But there were only 5 vertices in the bipyramid—not enough. So there is no Hamiltonian cycle.

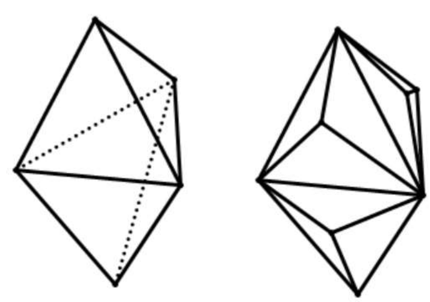

Figure 10. A triangular bipyramid with pyramids erected on each face.

S: That is a surprising and elegant example and argument. We could have a boring (non-baseball) seam with the three edges of the “equator.” Do you think we will encounter any more challenges?

J: Yes! Even more can go wrong. Did you know there are un-unfoldable polyhedra? That is, you can’t cut up the surface into one piece along edges and lay it out without some faces overlapping. Figure 11 illustrates one among many I found in Demaine and O’Rourke [2, 310].

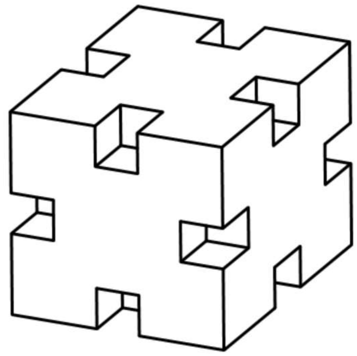

Figure 11 An un-unfoldable polyhedron.

S: That is pretty amazing, but we get to split the surface into two pieces. Maybe these examples with Hamiltonian cycles might fold out without overlapping provided we have two pieces.

J: This example doesn't have a Hamiltonian cycle because each edge of the big cube has one of these little cubes cut out. (The reasoning is like the bipyramid with attached pyramids.)

S: But we could just have little cubes cut out of the eight edges we use in a baseball seam of a cube, as in Figure 12. Then I can find a Hamiltonian cycle. For each of the indented cubes we can use seven of the eight edges of a baseball seam.

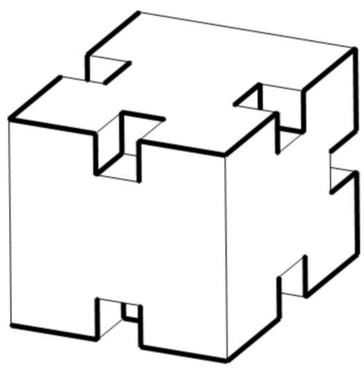

Figure 12. An indented cube with a Hamiltonian cycle.

J: Impressive. But when we unfold the piece with the top square, the two flaps hanging down from the cut-out square would fill up the same spot, where there is a missing small square. So even in two pieces, this is un-unfoldable.

S: Oh! I was hoping having two pieces might avoid that.

J: Hold on a minute, there is at least one other way of selecting seven edges of the indented cube, as in Figure 13.

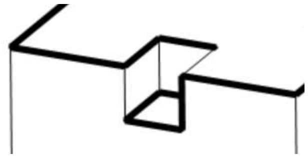

Figure 13 An alternative way to make a
Hamiltonian cycle on the indented cube.

S: I see.  But when we unfold the piece with the top square, there are two little squares that need to come up and the second one overlaps with part of the bigger square.  I guess no matter how we make out Hamiltonian cycle, this polyhedron doesn't have a baseball seam.

J: It seems that characterizing polyhedra that have baseball seams is still an open question, especially since we know now that having a Hamiltonian cycle does not guarantee a baseball seam.

S: I just remembered that in regard to Euler's formula, our polyhedron needs to have no holes. What happens if there is a hole?

J: You mean like a box with a hole cut through it? I've drawn a simplified version (figure 14) with twelve vertices and twelve faces.  We could have the seam separate the top four faces and two opposite sides (labelled S1 and S3) from the bottom four faces and the other two opposite sides.

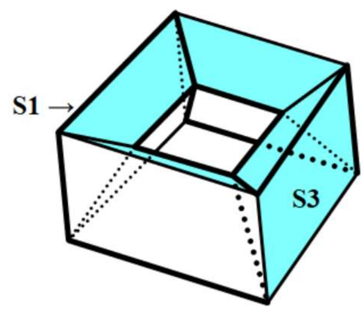


Figure 14. A polyhedron with a hole.

S: But there are two parts to the seam: the four edges on the inside and the eight that go around the outside like a $90^\circ$ rotation of the baseball seam of the cube in the right half of figure 2.

J: Oh! and the four top faces and the four bottom faces don't fold flat. So, that is definitely not our seam. Instead, we'd need two cuts, one between two faces on the top and one between two faces on the bottom to get them to fold flat.

S: At least with these two extra cuts the two parts to the seam would be joined.  But we now have more edges in the seam than vertices in the polyhedron.  So it isn't a Hamiltonian cycle.

J: Here (Figure 15) is an alternative split of the faces.  I shaded the six faces that form one piece. The set of edges separating those shaded faces from the other faces goes through each vertex and so ought to use up twelve edges. I count a total of twenty-four edges. The six faces of one piece should have five edges connecting them, but that only adds to $12 + 5 + 5 = 22$. What am I missing?

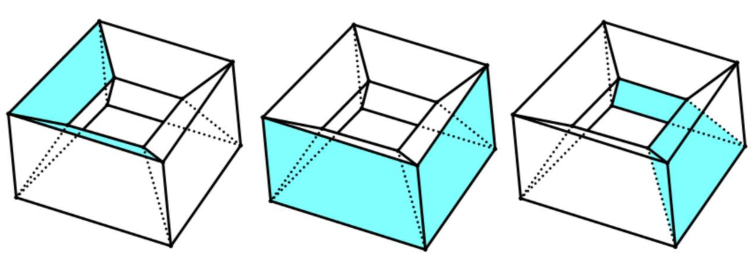

Figure 15. Six colored faces of a polyhedron with a hole.

S: I think there are actually 14 edges separating the two pieces, just as in the other dissection. It is a bit hard to count with this drawing, but I made a three-dimensional model and counted them. Euler's formula for polyhedra with holes has an extra term: $V - E + F = 2 - 2H$, where $H$ is the number of holes. (See West [3, 287]) Here $H = 1$, $V = 12 = F$ and $E = 24$. No matter how we cut the faces into two congruent pieces, each piece will have six faces and five edges connecting them. That leaves fourteen edges to form the seam.

J: There seems plenty more to explore and lots of directions to go. I love how we keep coming up with more questions to explore together.

S: This shows one of the cool things about math. We can start with intuitive ideas and through reasoning and mathematical creativity we can find ourselves in places we couldn't have imagined.

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3. Douglas West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.