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How Effective Is the Efficiency Gap?
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Gerrymandering has affected U. S. politics since at least 1812. A political cartoon that year decried this tactic by then Massachusetts Governor Elbridge Gerry. (Gerrymandering is manipulating the boundaries of districts to unfairly benefit a group.)

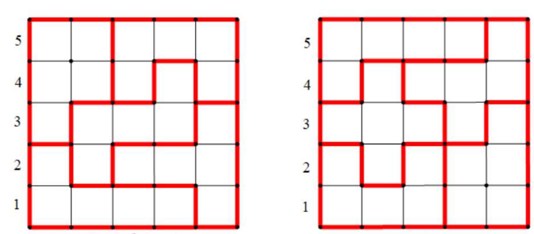
While we may feel we know a gerrymander when we see one, finding a meaningful metric has proven challenging. This article uses elementary mathematics to investigate the efficiency gap, a recent model proposed to measure gerrymandering. This measure came in response to a Supreme Court opinion by Justice Anthony Kennedy in 2004. In Vieth v. Jubelirer, Kennedy appeared open to "judicial relief if some limited and precise rationale were found to correct an established violation of the Constitution in some redistricting cases." [1]

Politicians have an incentive to draw district lines to their own advantage, maximizing the number of districts their party can win. "Packing" refers to a heavy concentration of votes from another party into a few districts to limit that party's representation. "Cracking" by one party denotes the allocation of another party's voters to ensure that the other party has a minority in as many districts as possible.

In their 2015 paper, Nicholas O. Stephanopoulos and Eric M. McGhee proposed the efficiency gap, a model which they believed could measure gerrymandering, enabling courts to make clean decisions: "It captures, in a single tidy number, all of the packing and cracking decisions that go into a district plan… We propose setting thresholds above which plans would be presumptively unconstitutional…" [2, 831]

We'll define the efficiency gap after setting the stage with an extended example.

Example 1. A "state" has twenty-five voters, represented as small squares in a 5×5 array, to divide into five districts of five voters each. We require the five squares of a district to be connected edge to edge to mimic real districts in the U. S. A. Figures 1 and 2 give two possible drawings of districts.



Figures 1 and 2. Possible district plans.

We'll split the twenty-five voters into two political parties, purple with nine voters and green with the remaining sixteen. The purple could just barely have a majority in three districts. At the other extreme, it is possible for the green party to have a majority in all five districts. Figure 3 shows two specific but randomly chosen placements of the nine purple voters and sixteen green voters in the district plans of Figures 1 and 2.

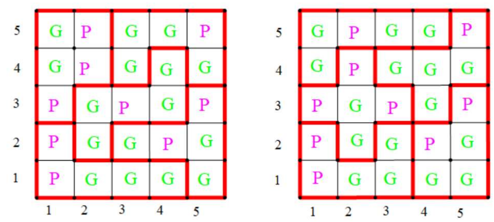


Figure 3. The placement of voters in the district plans.

In simulations, the green party occasionally won three or five districts, but far more often four districts. (The theoretical probability use multinomial coefficients. For green to win two districts and purple three, the probability is $\frac{\binom{16}{5\,5\,2\,2\,2}\binom{9}{0\,0\,3\,3\,3}}{\binom{25}{5\,5\,5\,5\,5}} = 0.005$. More complex computations show that the probabilities for three, four and five districts are 0.263, 0.610 and 0.122, respectively.)

Given the particular placement of the purple and green voters in Figure 3, it is possible to gerrymander the districts so that the purple party wins three districts, although I could only find three ways to achieve this. The three districts have three purple voters and two green voters. The remaining ten green voters must be "packed" in the other two districts, as in Figure 4. It is much easier to draw district lines, as in Figure 5, enabling green to win all five districts by "cracking" the purple voters.

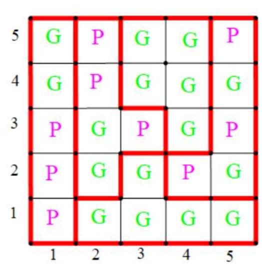


Figure 4. Gerrymandering for purple

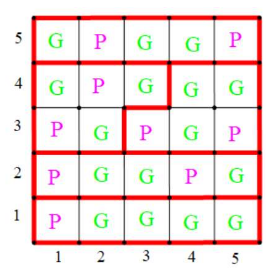


Figure 5. Gerrymandering for green

Stephanopoulos and McGhee's measure looks at the imbalance of what they termed wasted votes. They are assuming there are just two parties. They defined all the losing votes in a district as "wasted" as well as those winning votes in excess of what was needed to win. For instance, in an election with 100 votes, 51 are needed to win. The other 49 votes are wasted in their counting, but they pay attention to how these wasted votes are split between the winner and loser. For the districts in Example 1, three votes are needed to win and so two are wasted in each district, giving a total of ten wasted votes. The efficiency gap compares the total wasted votes of each party. They consider a big imbalance between the two party's wasted votes as indicative of packing and/or cracking and so possible gerrymandering. Our examples use an odd number of voters per district to avoid the issue of ties. Following Stephanopoulos and McGhee, we restrict our attention to the case of two parties.

Definition. If a party loses the election in a district, all of its votes in that district are **wasted**. If a party wins the election in a district, all of its votes in that district in excess of a bare majority are **wasted**. Let W_k be the wasted votes for party k from all districts and V the total number of votes. The **efficiency gap** is $E = \frac{W_1 - W_2}{V}$. (Positive efficiency gaps indicate an advantage for party 2, negative values for party 1.)

In Example 1 denote green as party 1 and purple as 2. There are ten wasted votes. In Figure 4 $E = \frac{10-0}{25} = 0.4$, which is as positive as possible. In Figure 5 $E = \frac{1-9}{25} = -0.32$ is as negative as possible, given that purple has at most nine votes to waste. In contrast the efficiency gaps for the districts of Figure 3 are both $E = \frac{4-6}{25} = -0.08$, mildly favoring green. A design in which green wins three districts will give an efficiency gap of $\frac{7-3}{25} = 0.16$.

These examples suggest that large efficiency gaps, whether positive or negative, may indicate gerrymandering and values close to zero perhaps more unbiased district lines. That is what the developers of the efficiency gap wanted to accomplish. Let's look more deeply.

Example 2. Table 1 gives a possible split for 9 districts of nine votes each.

District	A	B	C	D	E	F	G	H	I
Green	7	7	7	7	7	7	7	7	7
Purple	2	2	2	2	2	2	2	2	2

Table 1.

The efficiency gap is a perfect $\frac{18-18}{81} = 0$, but purple with $\frac{2}{9}$ of the votes gets no districts. We might suspect green of a uniform cracking of purple voters, but the efficiency gap would not detect anything amiss.

It might seem more fair for purple to win one or even two districts out of the nine since $\frac{2}{9}$ of the districts would be proportional. However, Table 2, where purple wins one district, gives $E = \frac{23-13}{81} = 0.123$. Table 3, where purple wins two districts, which might seem fairest, gives $E = \frac{28-8}{81} = 0.247$. With purely numerical values we can't attribute the intention of packing or cracking. But these examples raise the question of how well the efficiency gap measures what the authors intended with regard to gerrymandering or fairness.

District	A	B	C	D	E	F	G	H	I
Green	4	8	8	8	7	7	7	7	7
Purple	5	1	1	1	2	2	2	2	2

Table 2.

District	A	B	C	D	E	F	G	H	I
Green	4	4	7	8	8	8	8	8	8
Purple	5	5	2	1	1	1	1	1	1

Table 3.

Example 3. In a very evenly divided electorate, small shifts can swing the efficiency gap. Consider the scenarios of Tables 4 and 5.

	A	B	C	D	E	F
Green	3	3	3	2	2	2
Purple	2	2	2	3	3	3

Table 4.

	A	B	C	D	E	F
Green	3	3	2	2	2	2
Purple	2	2	3	3	3	3

Table 5.

In Table 4, the efficiency gap is 0, whereas in Table 5 a shift of one vote in district C gives an efficiency gap of $\frac{4}{30} = 0.133$. If districts A and B successively have the same shifts, the efficiency gap grows to $\frac{8}{30} = 0.267$ and $\frac{12}{30} = 0.400$, the maximum gap possible with this many votes per district.

These examples suggest that the efficiency gap doesn't measure suspected packing and cracking of districts as well as its creators had hoped. In fact, assuming districts all have the same number of votes, we will see that the efficiency gap depends not on their distribution in districts, but only on (a) the number of districts each party wins, (b) the number of votes per district, and (c) the total number of votes of each party. Table 6 illustrates this idea concretely with 10 districts, each with 21 votes. The column indicates the percent of the 210 votes that party 1 has and the row the number of districts party 1

wins. The values in the table are the efficiency gaps, where positive values favor party 2. An entry of NA indicates that it is impossible for a party with that percentage of votes to win that number of districts. Consider for instance, the efficiency gap entry 0.410 in the row for party 1 winning only 3 districts in the column where that party has 60% of the total votes or 126 votes, while party 2 has 84 votes. To win 3 districts, party 1 needed $11 \times 3 = 33$ votes, meaning $126 - 33 = 93$ were wasted. With party 2 winning 7 districts, it wastes only $84 - 11 \times 7 = 7$ votes. Thus, the efficiency gap is $\frac{93-7}{210} = 0.4095\ldots$ Note that we didn't need to know in which districts the wasted votes appear.

	50	60	70	80	90
1	0.419	NA	NA	NA	NA
2	0.314	NA	NA	NA	NA
3	0.210	0.410	NA	NA	NA
4	0.105	0.305	NA	NA	NA
5	0	0.200	0.400	NA	NA
6	-0.105	0.095	0.295	NA	NA
7	-0.210	-0.010	0.190	0.390	NA
8	-0.314	-0.114	0.086	0.286	NA
9	-0.419	-0.219	-0.019	0.181	0.381
10	NA	-0.324	-0.124	0.076	0.276

Table 6. Examples of Efficiency Gaps.

When each party has 50% of the votes, the efficiency gap matches our intuition about fairness: the more unequal the distribution of wins, the further the efficiency gap is from zero. Indeed to create a distribution of votes where a party has 50% of the vote but wins only one district in the scenario of Table 6 requires what looks like careful packing and cracking.

The situation changes when a party has noticeably more than 50%, as Bernstein and Duchin point out in [3, 1022]. The table illustrates that the efficiency gap closest to zero occurs when party 1 wins an even larger percentage of the districts than its percentage of overall votes. We will use some algebra to show that the percentage of winning districts over 50% should be double the percentage of votes over 50%.

Let each of D districts have $2n + 1$ votes, D_k be the number of districts party k wins and V_k that party's total votes. That is, $V = (2n + 1)D = V_1 + V_2$ and $D = D_1 + D_2$. Party k needs $D_k(n + 1)$ votes to win the D_k districts and so has $V_k - D_k(n + 1)$ wasted votes. Thus, the efficiency gap is

$$E = \frac{V_1 - D_1(n+1) - (V_2 - D_2(n+1))}{(2n+1)D}$$

$$= \frac{V_1 - V_2 - (n+1)(D_1 - D_2)}{V} = \frac{V_1 - V_2}{V} - \frac{n+1}{2n+1}\left(\frac{D_1 - D_2}{D}\right).$$

Let's examine the parts of this value. The value $\frac{V_k}{V}$ is the overall proportion of votes for party k and $\frac{V_1 - V_2}{V}$ is the difference in these proportions. If party 1 has 60% of the votes to 40% for party 2, $\frac{V_1 - V_2}{V} = 0.2$. Similarly, $\frac{D_1 - D_2}{D}$ is the difference of the proportions of the districts each party won. For large n, $\frac{n+1}{2n+1}$ is effectively $\frac{1}{2}$ and $\frac{n+1}{2n+1}\left(\frac{D_1 - D_2}{D}\right)$ thus is half the difference $\frac{D_1 - D_2}{D}$. If party 1 has 70% of the seats to 30% for party 2, $\frac{n+1}{2n+1}\left(\frac{D_1 - D_2}{D}\right) \approx \frac{1}{2}(0.4) = 0.2$. Because of this factor of one half, we see to make the efficiency gap close to zero the larger party should win disproportionately many districts. By the time party 1 has 75% of the votes, $\frac{V_1 - V_2}{V} = 0.5$. For $\frac{n+1}{2n+1}\left(\frac{D_1 - D_2}{D}\right)$ to equal that, party 1 should win all the districts. (Example 2 illustrates this.)

The preceding examples and discussion indicate that the efficiency gap doesn't capture all that its authors had intended. But the issue of gerrymandering is even more complicated. Real districts need to consider multiple factors, including geographical and, due to the Voting Rights Act, racial and ethnic factors. Clearly the efficiency gap can't take these into account. Indeed, the authors of this model emphasize that their measure would only be a first step in the process. [2, 898]

In Whitford v. Gill a panel of a federal district court in 2016 used the efficiency gap as a major reason for declaring unconstitutional the district lines previously drawn by the Wisconsin legislature. In two other gerrymandering cases, the U. S. Supreme Court decided in 2019 that partisan gerrymandering was a political question, not one for the federal courts. This vacated the earlier ruling about Wisconsin's districts. (State courts can declare partisan gerrymandering unconstitutional. The Pennsylvania Supreme Court did so in 2018. Racial gerrymandering is still illegal.)

Moon Duchin suggests a possibly better way to approach the issue of gerrymandering of a proposed redistricting plan using simulations and probability. If in a suitably random selection of redistricting plans the proposed plan appears an extreme outlier, then one could investigate the plan for potential gerrymandering using other factors. [4]

Our investigation of the efficiency gap suggests that it may be useful in quantifying gerrymandering when the electorate is close to evenly split, but otherwise has some weaknesses and limitations. First, when one party has a significantly larger share of the overall votes, the efficiency gap is weighted doubly towards that party. Also, a small shift of votes in a close election can make a disproportionate change in the efficiency gap. Further, it doesn't look at gerrymandering—the actual distribution of votes in districts, but rather the total votes of each party and the number of districts each wins. Finally, real districts need to consider other relevant aspects, including geographical, racial and ethnic factors.

"[G]errymandering is a fundamentally multidimensional problem, so it is manifestly impossible to convert that into a single number without a loss of information that is bound to produce many false positives or false negatives for gerrymandering." – Moon Duchin

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