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## The possibility of impossible pyramids

Thomas Q. Sibley

College of Saint Benedict/Saint John's University, [tsibley@csbsju.edu](mailto:tsibley@csbsju.edu)

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# The Possibility of Impossible Pyramids

TOM SIBLEY  
 St. John's University  
 Collegeville, MN 56321

## Introduction

When can we form a triangle or a pyramid from edges with given lengths? Euclid found that for three segments to be the sides of a triangle, it is necessary and sufficient for the length of each segment to be shorter than the sum of the lengths of the other two. Thus, for any three lengths that could potentially be sides of a triangle, there actually is a Euclidean triangle whose sides have those lengths. In short, Euclidean plane geometry is “triangle complete,” as defined and shown analytically below. It seems natural to expect that Euclidean space is “pyramid complete”; that is, given six lengths with appropriate restrictions we can find a Euclidean pyramid whose edges have those lengths. But could there be “impossible pyramids”? Might there be lengths for the sides of a pyramid that form four triangular faces even though no Euclidean pyramid exists with those lengths?

As part of this investigation we shall describe a family of metric spaces which includes that of Euclidean geometry.

DEFINITIONS. A metric space  $(X, d)$  is a non-empty set  $X$  together with a metric (or distance)  $d$  such that for any elements  $P, Q, R \in X$ ,

- (i)  $d(P, Q) \geq 0$ ;
- (ii)  $d(P, Q) = d(Q, P)$ ;
- (iii)  $d(P, Q) = 0 \Leftrightarrow P = Q$ ;
- (iv)  $d(P, Q) + d(Q, R) \geq d(P, R)$ .

Property (iv) is the *triangle inequality*. A metric space is *triangle complete* if, for any three non-negative real numbers  $a, b$ , and  $c$  satisfying the triangle inequality in any order (i.e.,  $a \leq b + c$ ,  $b \leq a + c$ , and  $c \leq a + b$ ), there are three points  $P, Q$ , and  $R$  in  $X$  such that  $d(P, Q) = a$ ,  $d(P, R) = b$ , and  $d(Q, R) = c$ .

Let's show that Euclidean plane geometry (i.e.,  $\mathbb{R}^2$  with the usual Euclidean metric) is triangle complete. Given three non-negative reals  $a, b$ , and  $c$  satisfying the three triangle inequalities, let  $P = (0, 0)$  and  $Q = (a, 0)$ . We need to find a point  $R = (x, y)$  such that  $d(P, R) = b$  and  $d(Q, R) = c$  or, equivalently,  $x^2 + y^2 = b^2$  and  $(a - x)^2 + y^2 = c^2$ . These equations' solutions are  $x = (a^2 + b^2 - c^2)/2a$  and  $y = \pm \sqrt{b^2 - x^2} = \pm \sqrt{(b - x)(b + x)}$ . We will establish triangle completeness once we show that  $b^2 - x^2 \geq 0$ . Now  $b + x = ((a + b)^2 - c^2)/2a \geq 0$  because  $a + b \geq c$ . The two other triangle inequalities show that  $|a - b| \leq c$  and so  $b - x = (c^2 - (a - b)^2)/2a \geq 0$ .

The hyperbolic plane is also triangle complete since Euclid's Proposition 22 from Book I holds in hyperbolic geometry. However, not every metric space is triangle complete. For instance, a sphere in  $\mathbb{R}^3$  of radius  $r$  with distances measured along its surface cannot be triangle complete because we cannot find points arbitrarily far apart on a sphere. In fact, even if we restrict  $a, b$ , and  $c$  to distances that occur on the sphere and satisfy the triangle inequality, they still need not determine a triangle

whose sides are arcs of great circles. For example, first consider an equilateral triangle on a sphere whose three vertices lie on the equator. Note that no equilateral triangle on the sphere can have longer sides. Let  $d = 2\pi r/3$  be the distance between two of these points and let  $a$ ,  $b$ , and  $c$  be equal lengths slightly larger than  $d$ . Then there is no spherical triangle with sides  $a$ ,  $b$ , and  $c$ .

## Euclidean pyramids

Let's turn to the problem of constructing a pyramid in three-dimensional Euclidean space from six given lengths. First of all, appropriate triples of lengths must form four triangles to make the faces of the pyramid, say  $\triangle PQR$ ,  $\triangle PQS$ ,  $\triangle PRS$ , and  $\triangle QRS$ . Thus the triangle inequalities must hold for these triples. We want to glue the corresponding edges of the triangles together to make a pyramid. We formalize this intuitive idea as follows:

**DEFINITION.** A metric space  $(X, d)$  is pyramid complete if, for any sextuple of non-negative real numbers  $(a, b, c, d, e, f)$  such that each of the triples  $\{a, b, c\}$ ,  $\{a, d, e\}$ ,  $\{b, d, f\}$ , and  $\{c, e, f\}$  satisfies the triangle inequalities in any order, there are four points  $P$ ,  $Q$ ,  $R$ , and  $S$  in  $X$  so that  $d(P, Q) = a$ ,  $d(P, R) = b$ ,  $d(Q, R) = c$ ,  $d(P, S) = d$ ,  $d(Q, S) = e$ , and  $d(R, S) = f$ .

(See FIGURE 1; the triple  $\{a, b, c\}$  corresponds to  $\triangle PQR$ .)

Is Euclidean space pyramid complete? The following example shows that the answer is no.

*Example 1.* Consider the sextuple  $(14, 8, 8, 8, 8, 8)$ . We need  $d(P, Q) = 14$ , and all the others to be 8. We can readily make  $d(P, R) = d(P, S) = d(Q, R) = d(Q, S) = 8$  as well, but we will see that these distances restrict the possible values of  $d(R, S)$ . If we fix  $P$ ,  $Q$ , and  $R$  and rotate triangle  $\triangle PSQ$  around the side  $PQ$ , we get the largest distance between  $R$  and  $S$  when all four points are in the same plane; see FIGURE 2. For  $M$ , the midpoint of  $PQ$ , the Pythagorean theorem applied to  $\triangle PMS$  and  $\triangle PMR$  gives  $d(S, M) = d(R, M) = \sqrt{d(P, R)^2 - d(P, M)^2} = \sqrt{8^2 - 7^2} \approx 3.873$ . Thus, no matter how we rotate  $\triangle PSQ$ ,  $d(R, S) \leq d(R, M) + d(M, S) < 7.75$ , which is less than the required 8.

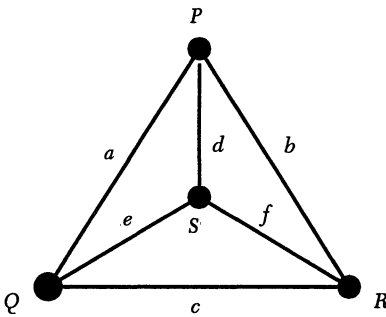


FIGURE 1

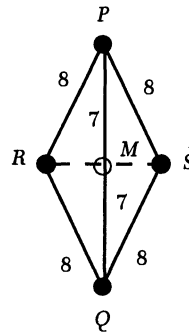


FIGURE 2

**Remarks** A scaled down version of Example 1 shows that no 3-dimensional space that is “locally Euclidean” (technically a 3-dimensional manifold), such as hyperbolic space, can be pyramid complete. Several mathematicians, starting with Niccolo Tartaglia in 1560, have published versions of the following formula relating the volume  $V$  of a pyramid to the lengths of its six sides:

$$V^2 = \frac{1}{6}(a^2f^2(b^2 + c^2 + e^2 + d^2) + b^2e^2(a^2 + c^2 + f^2 + d^2) + c^2d^2(a^2 + b^2 + f^2 + e^2) - a^4f^2 - a^2f^4 - b^4e^2 - b^2e^4 - c^4d^2 - c^2d^4 - a^2b^2c^2 - a^2e^2d^2 - b^2f^2d^2 - c^2f^2e^2).$$

The triangle inequalities for the faces together with the requirement that  $V^2$  non-negative give necessary and sufficient conditions on the lengths for a Euclidean pyramid to exist (see [4]).

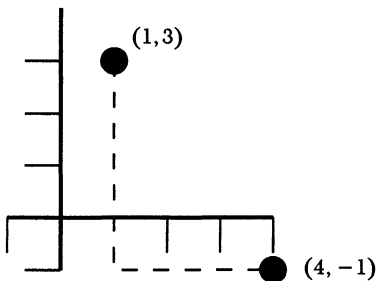
### The taxicab metric

Are any metric spaces pyramid complete? Yes! We show that  $\mathbb{R}^3$  with the taxicab metric is pyramid complete. The taxicab metric provides an easily-explored geometry, often called taxicab geometry. (See, e.g., [2], [3], [5], and [7].) The “taxicab” name comes, in  $\mathbb{R}^2$ , from the distances travelled by cars if all streets run either North-South or East-West. (See FIGURE 3.) Cars that stay on the roads cannot benefit from the Pythagorean theorem, so the total distance between any two points is the sum of the distances in the principal directions. We consider the taxicab metric only in  $\mathbb{R}^3$ , although it is readily defined in  $\mathbb{R}^n$ .

**DEFINITION.** For two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  in  $\mathbb{R}^3$ ,

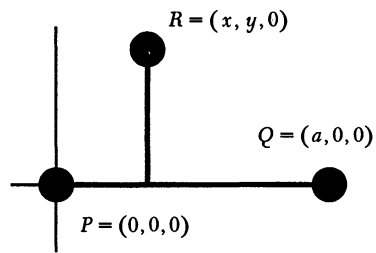
$$d_T((x_1, y_1, z_1), (x_2, y_2, z_2)) = |x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|.$$

See, e.g., [6, pp. 214–219] for a proof that  $\mathbb{R}^3$  with the taxicab metric  $d_T$  is a metric space. Moreover, we can show that  $\mathbb{R}^3$  with the taxicab metric is triangle complete. Let  $a, b$ , and  $c$  satisfy the three triangle inequalities and, without loss of generality, assume that  $a$  is the largest. Let  $P = (0, 0, 0)$ ,  $Q = (a, 0, 0)$ , and  $R = (x, y, 0)$ , where  $x = (a + b - c)/2$  and  $y = (b + c - a)/2$ . From the triangle inequalities  $0 \leq x$  and  $0 \leq y$ . It is easy to check that  $d_T(P, R) = x + y = b$  and  $d_T(Q, R) = (a - x) + y = c$ , as in FIGURE 4. Note that this argument makes no use of the third coordinate, so the “taxicab plane” is also triangle complete.



**FIGURE 3**

In the taxicab metric  
 $d((1,3),(4,-1)) = 3 + 4 = 7$



**FIGURE 4**



From the definition of the taxicab metric we get the equations

$$t + u = a, \quad t + v = b, \quad \text{and} \quad u + v = c.$$

The solutions

$$t = .5(a + b - c), \quad u = .5(a + c - b), \quad \text{and} \quad v = .5(b + c - a)$$

are all non-negative because of the triangle inequalities satisfied by  $a$ ,  $b$ , and  $c$ .

Next we place  $S$ . We could solve three additional equations for  $S$  with various cases, but a geometric approach provides more insight. As in the discussion following Example 2 we treat  $d_T(R, S)$  as a variable. We use the fourth and fifth numbers,  $d$  and  $e$ , to determine the range of possible locations for  $S$  relative to points  $P$  and  $Q$ , which is the range of values for  $d_T(R, S)$ . We will show that this range includes  $f$ . Figures 7 through 10 illustrate our strategy. The dashed lines in FIGURES 7, 8, and 9 describe the possibilities for  $S$  when  $b \geq c$ . If  $b < c$ , only FIGURE 8 changes significantly, as shown in FIGURE 10. In all cases, we will show that  $S'$  is the candidate for  $S$  as close to  $R$  as possible and  $S^*$  is the candidate for  $S$  as far from  $R$  as possible.

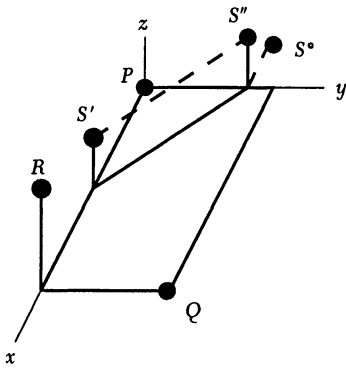


FIGURE 7

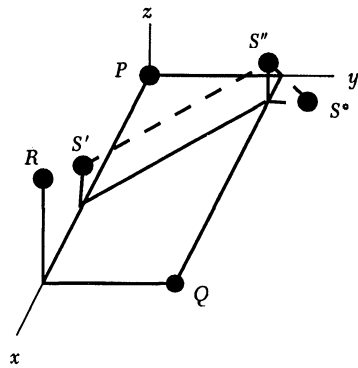


FIGURE 8

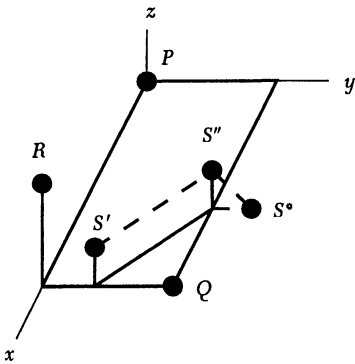


FIGURE 9

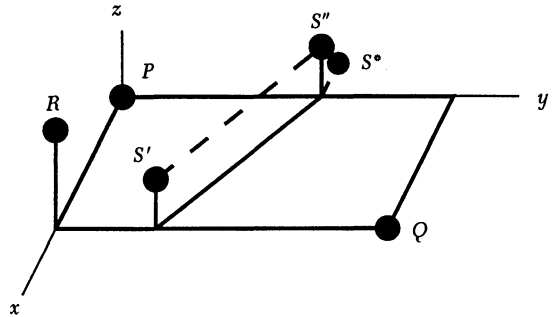


FIGURE 10

We consider now the possibilities for  $S$ , showing that these points satisfy  $d = d_T(P, S)$  and  $e = d_T(Q, S)$ . Let  $k = 0.5(d + e - a)$ . Because  $a$ ,  $d$ , and  $e$  satisfy the triangle inequality in any order, we see that  $k \geq 0$ . As in Example 2,  $k$  will be the height above the  $xy$ -plane of the points on the dashed line between  $S'$  and  $S''$ . Consider the points  $X = (x, y, 0)$  satisfying  $0 \leq x$ ,  $y \leq d - k = 0.5(a + d - e)$  and  $x + y = d - k$ . These points  $X$  form the solid diagonal line. Then  $d_T(P, X) = d - k$  and  $d_T(Q, X) = e - k = 0.5(a + e - d)$ , since the triangle inequalities for  $a$ ,  $d$ , and  $e$  ensure that both  $d - k \geq 0$  and  $e - k \geq 0$ . Given  $X = (x, y, 0)$  as above, the points  $S = (x, y, k)$  satisfy  $d_T(P, S) = d - k + k = d$  and  $d_T(Q, S) = e$ . The tables below give the coordinates for  $S'$ ,  $S''$ ,  $S^*$ , and  $S$  between  $S''$  and  $S^*$  for the cases corresponding to FIGURE 7 through 10, which depend on how  $d - k$  compares with  $t$  and  $u$ . For the column for  $S$ ,  $0 \leq w \leq k$ .

Figure	$d - k$	$S'$	$S''$
7	$d - k \leq t, u$	$(d - k, 0, k)$	$(0, d - k, k)$
8	$u < d - k \leq t$	$(d - k, 0, k)$	$(d - k - u, u, k)$
9	$t, u < d - k$	$(t, d - k - t, k)$	$(d - k - u, u, k)$
10	$t < d - k \leq u$	$(t, d - k - t, k)$	$(0, d - k, k)$

Figure	$S^*$	$S$ between $S''$ and $S^*$
7	$(-k, d - k, 0)$	$(-w, d - k, k - w)$
8	$(d - k - u, u + k, 0)$	$(d - k - u, u + w, k - w)$
9	$(d - k - u, u + k, 0)$	$(d - k - u, u + w, k - w)$
10	$(-k, d - k, 0)$	$(-w, d - k, k - w)$

We can readily check that, throughout the table,

$$d = d_T(P, S') = d_T(P, S'') = d_T(P, S^*) = d_T(P, S).$$

Similarly, we find that, throughout the table,

$$d_T(Q, S') = d_T(Q, S'') = d_T(Q, S^*) = d_T(Q, S) = t + u + 2k - d,$$

which equals  $e$  once we rewrite  $t$ ,  $u$ , and  $k$  in terms of  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $e$ . Thus in all cases the points between  $S'$  and  $S''$  and between  $S''$  and  $S^*$  are the correct distances from  $P$  and  $Q$ .

Finally, we show that the points  $S'$  and  $S^*$  in each case give, respectively, the minimum and maximum distances between  $R$  and  $S$  compatible with the relevant triangle inequalities. We now need the assumption  $b + c \geq d + e$ , which implies that  $d_T(P, R) + d_T(Q, R)$  exceeds  $d_T(P, Q)$  by more than  $d_T(P, S) + d_T(Q, S)$  exceeds  $d_T(P, Q)$ . Hence the  $z$ -coordinate of  $R$  must be greater than the  $z$ -coordinate of  $S$ . In FIGURES 7 and 8, where  $S' = (d - k, 0, k)$ ,  $S'$  is “between”  $P$  and  $R$ . That is,  $d_T(P, R) = d_T(P, S') + d_T(S', R)$ . Because  $d_T(P, R) = b$ ,  $d_T(P, S') = d$  and  $b \leq d + f$ , we see that  $d_T(S', R) \leq f$ . In FIGURES 9 and 10,  $S' = (d - k - u, u, k)$  is “between”  $Q$  and  $R$  and a similar argument shows that  $d_T(S', R) \leq f$ . In the same way we see that  $f \leq b + d = d_T(R, S^*)$  in FIGURES 7 and 10 and  $f \leq c + e = d_T(R, S^*)$  in FIGURES 8 and 9. In short,  $d_T(R, S') \leq f \leq d_T(R, S^*)$ . This completes the proof. ■

### Related metrics

The Euclidean and taxicab metrics are special cases of the so-called  $p$ -metrics (for  $1 \leq p < \infty$ ), a family of metrics defined as follows for  $\mathbb{R}^3$ .

DEFINITION. On  $\mathbb{R}^3$ , define  $d_p$  for any real number  $p \geq 1$  by

$$d_p((x, y, z), (k, l, m)) = (|x - k|^p + |y - l|^p + |z - m|^p)^{1/p}.$$

The Euclidean metric is the case  $p = 2$  and the taxicab metric is the case  $p = 1$ . See, e.g., [6, pp. 214–219] for a proof that  $d_p$  is a metric on  $\mathbb{R}^n$ .

It can be shown that  $\mathbb{R}^n$  with any of the  $p$ -metrics is triangle complete. Since the proof requires more advanced ideas from real analysis, we omit it. Interestingly, however, we can settle the question of pyramid completeness without advanced mathematics. FIGURE 11 illustrates in two dimensions the shape of the “ $p$ -circles” of the same radius for different values of  $p < \infty$ . Note that the taxicab “circle” has straight sides, while all the others are curved. This flatness is the key to pyramid completeness for the taxicab metric. Each of the other  $p$ -circles is *strictly convex*: Each point on these circles has a tangent line that intersects the circle in only that point. Similarly, for  $p > 1$  the three-dimensional “ $p$ -spheres” are strictly convex: Each point has a tangent plane that intersects the  $p$ -sphere at only one point.

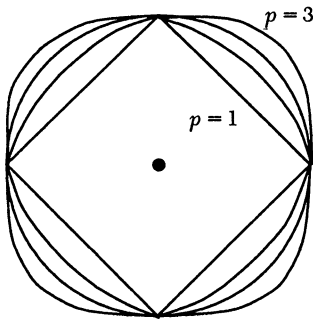


FIGURE 11  
 $p$  circles for  
 $p = 1, p = 1.5, p = 2, p = 3$ .

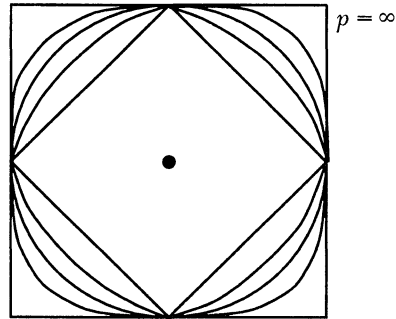


FIGURE 12  
 $p$  circles  
including  $p = \infty$

EXAMPLE 3. The metric space  $\mathbb{R}^3$  with the  $p$ -metric for  $1 < p < \infty$  is not pyramid complete. Consider the following values for the sides of the pyramid:  $a = 10, b = c = d = e = 5$  and  $f > 0$ . Let  $P$  and  $Q$  be any two points in  $\mathbb{R}^3$  with  $d_p(P, Q) = 10$ . Consider the  $p$ -spheres of radius 5 centered at  $P$  and  $Q$ . By the strict convexity of these  $p$ -spheres, they have exactly one point of intersection. This means that the only possibility for a pyramid with the given sides  $a, b, c, d,$  and  $e$  is for  $f = 0$ . Hence  $\mathbb{R}^3$  with the  $p$ -metric is not pyramid complete for  $1 < p < \infty$ .

REMARK. A more sophisticated argument shows that for any particular value of  $p > 1$  and  $a > 0$  there are values for  $b, c, d, e,$  and  $f$  all the same and slightly bigger than  $a/2$  giving actual triangles that cannot be made into pyramids.

The family of  $p$ -metrics has one more member—the  $\infty$ -metric. The  $p$ -metrics arise in functional analysis, an area unrelated to triangle and pyramid completeness. (See [1] and [6].)



DEFINITION. On  $\mathbb{R}^3$ , define  $d_\infty$  by

$$d_\infty((x, y, z), (k, l, m)) = \max\{|x - k|, |y - l|, |z - m|\}.$$

FIGURE 12 adds the “ $\infty$ -circle” to the  $p$ -circles of FIGURE 11. As  $p$  increases, the shape of a  $p$ -circle approaches the shape of the  $\infty$ -circle; this indicates why the notation  $d_\infty$  is natural. More precisely, in  $\mathbb{R}^n$ ,  $d_\infty(P, Q) = \lim_{p \rightarrow \infty} d_p(P, Q)$ . (See, e.g., [6] for more on  $d_\infty$ .) Because circles in both the  $d_1$  and  $d_\infty$  metrics have straight sides, we conjecture that  $\mathbb{R}^3$  with the  $d_\infty$  metric is pyramid complete.

THEOREM 2. The metric space  $\mathbb{R}^3$  with  $d_\infty$ , the  $\infty$ -metric, is pyramid complete.

Proof. Suppose that for the sextuple of non-negative numbers  $(a, b, c, d, e, f)$ , all triangle inequalities hold for the triples  $\{a, b, c\}$ ,  $\{a, d, e\}$ ,  $\{b, d, f\}$ , and  $\{c, e, f\}$ . (See FIGURE 1 for the relationship between these lengths and the points  $P, Q, R$ , and  $S$ .) For ease we consider the case where  $a$  is the largest value and  $e$  is the smallest among  $b, c, d$ , and  $e$ . (The other cases require only a relabeling of points.) Let  $P = (b, 0, 0)$ ,  $Q = (c, a, e)$ ,  $R = (0, b, 0)$ , and  $S = (f, d, 0)$ . (See FIGURE 13.)

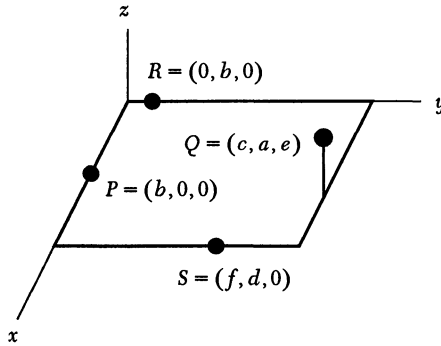


FIGURE 13

Then  $d_\infty(P, R) = b$ . Also,  $d_\infty(P, Q) = a$  and  $d_\infty(Q, R) = c$  because  $0 \leq e \leq c \leq a$  and the triangle inequality  $a \leq b + c$  guarantees  $a - b \leq c$ . The other triangle inequalities similarly ensure that  $d_\infty(P, S) = d$ ,  $d_\infty(Q, S) = e$  and  $d_\infty(R, S) = f$ . ■

EXAMPLE 4. An unordered set of six lengths may form a Euclidean pyramid in some orders, but not in others. Consider the sextuple  $(10, 7, 6, 6, 6, 6)$ . In this order there is a Euclidean pyramid with its vertices having approximate coordinates  $P = (10, 0, 0)$ ,  $Q = (0, 0, 0)$ ,  $R = (4.35, 4.132, 0)$ , and  $S = (5, -0.907, 3.190)$ . However, the reasoning in Example 1 shows that the sextuple  $(10, 6, 6, 6, 6, 7)$  fails to have a corresponding Euclidean pyramid. The definition of pyramid completeness we have chosen leads to simpler proofs than a weaker one allowing reordering of lengths. Examples 1 and 3 show that the  $p$ -metrics for  $1 < p < \infty$  are not pyramid complete, even allowing such reordering.

Interested readers can explore higher-dimensional or other variations of this problem, such as the following suggestion from Walter Sizer, of Moorhead State University:

The areas of the four faces of a pyramid satisfy an inequality similar to the triangle inequality: The sum of any three of these areas is not less than the fourth area. Given four numbers that satisfy this area inequality in all arrangements, is there a pyramid in Euclidean geometry (or other metric spaces) whose faces have these numbers as their areas?

In summary, only the metric spaces with the extreme metrics,  $p = 1$  and  $p = \infty$ , are pyramid complete.

**Acknowledgment.** I thank the referees for Example 4 and for other helpful suggestions.

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