Sublimital Analysis

Thomas Q. Sibley
College of Saint Benedict/Saint John's University, tsibley@csbsju.edu

Recommended Citation

Sublimital Analysis

Thomas Q. Sibley
St. John's University
Collegeville, MN 56321
tsibley@csbsju.edu

Limits of subsequences play a small supporting role in analysis. (See, for example, the Bolzano-Weierstrass Theorem.) However, in the typical undergraduate course we never seem to care what the limits actually are, suggesting that these "sublimits" might not deserve star billing. The article [3] by Zheng and Cheng in the references does consider such "sublimits" in a particular setting. This article takes a closer look at subsequences and their limits more generally. I owe a disclosure to those readers who connected the "sublimital" of the title with the word "subliminal." While "sublimits" may well be hidden in the original sequence, they aren't placed there to send subconscious messages. Instead we should think of them as enticements to mathematical exploration.

Example 1. Let (b_n) be the alternating sequence given by $b_n = (-1)^n(\frac{n-1}{n})$. The first few terms are $\frac{0}{1}$, $\frac{1}{2}$, $\frac{-2}{3}$, $\frac{3}{4}$, $\frac{-4}{5}$, ...

This sequence diverges, but it has subsequences converging to two "sublimits": $\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} \frac{2k-1}{2k} = 1$ and $\lim_{k\to\infty} b_{2k-1} = \lim_{k\to\infty} \frac{-(2k-2)}{2k-1} = -1$.

Definition. Given a sequence (a_n) of real numbers, a real number s is a *sublimit* of (a_n) if and only if there is some subsequence (a_{n_k}) of the original sequence such that $\lim_{k\to\infty} a_{n_k} = s$. Denote the set of all sublimits of a sequence (a_n) by $S(a_n)$.

Example 1 (continued). For the sequence (b_n), where $b_n = (-1)^n(\frac{n-1}{n})$ as given earlier, $S(b_n) = \{1, -1\}$. The reader is invited to show (b_n) has no other sublimits.

Remark. If a sequence (c_n) converges to a limit l, then $S(c_n) = \{l\}$ since l is the only possible sublimit.

Example 2. The sequence (d_n) given by $d_n = n$ has no sublimits and $S(d_n) = \varnothing$.

Example 3. The set Q of rationals is countable so there is a sequence (q_n) listing all of Q. Then every real number r is a sublimit of (q_n). To see this, note that for each $\varepsilon = \frac{1}{k} > 0$ there are infinitely many rationals in the open interval $(r - \varepsilon, r + \varepsilon)$. So for all $k \in$ N, we can choose $q_{n_k} \in$ Q such that $|q_{n_k} - r| < \frac{1}{k}$ and $n_k < n_{k+1}$. Thus the subsequence (q_{n_k}) converges to r and $S(q_n) =$ R.

In Example 3 each real number has its own subsequence, which the usual notation (q_{n_k}) can't indicate. The following notation overcomes that lack and will be useful in the proof of Theorem 1.

Definition. Given a sequence (a_n) and a set of its subsequences indexed by K, for $k \in K$ let $(a_n(k))$ denote the subsequence with index k and let $a_n(k, i)$ denote the ith term of $(a_n(k))$.

Example 3 naturally leads to the question "Given any set S of real numbers is there a sequence whose set of sublimits is S?" The answer, in a word, is "no." Our goal is to characterize the possible sets of sublimits. At the end we'll generalize this question to metric spaces.

In looking for ways to describe possible sets of sublimits we might well start with instances where subsequences appear in analysis courses. (See, for example, the text [1] by Abbott for definitions of terms used in this paragraph along with more on the theorems.) The Bolzano-Weierstrass theorem states that every bounded sequence has a convergent subsequence. So perhaps bounded sets play a role. However, the set of sublimits in Example 3 is definitely not bounded, so that property can't be part of the characterization of sets of sublimits. Another common role of subsequences is in the definition of sequentially compact. The Heine-Borel theorem informs us that sets are compact if and only if they are closed and bounded. Also, limits and sublimits are related to limit points, which appear in the definition of closed sets. So perhaps the characterization of sets of sublimits relates to closed sets. The sets of sublimits in Examples 1, 2 and 3 are, indeed, closed. Theorem 1 below confirms that all sets of sublimits are closed.

Theorem 1. Let (a_n) be any sequence of real numbers. Then $S(a_n)$, its set of sublimits, is a closed set.

Figure 1 explains the idea behind the proof. The right hand column is a sequence (s_k) of sublimits of the sequence (a_n) and its limit L. Each of the sublimits s_k has a subsequence $(a_n(k))$ converging to s_k. We need to create a new subsequence $(a_n(L))$ converging to L. Figure 1 suggests choosing the "diagonal" subsequence $(a_n(k, k))$. The reader is invited to construct an example where such a diagonal subsequence fails to converge to L. In the proof we shall generate a more sophisticated subsequence using the Axiom of Choice.

$$
\begin{array}{llll}
a_n(1,1) & a_n(1,2) & a_n(1,3) & \ldots & s_1 \\
a_n(2,1) & a_n(2,2) & a_n(2,3) & \ldots & s_2 \\
a_n(3,1) & a_n(3,2) & a_n(3,3) & \ldots & s_3 \\
\vdots & \vdots & \vdots & & \vdots \\
a_n(k,1) & a_n(k,2) & \ldots & a_n(k,k) & \ldots & s_k \\
\vdots & \vdots & & \ddots & \vdots \\
& & & & L
\end{array}
$$

Figure 1.

Proof of Theorem 1. Let (a_n) be any sequence and, in order to show $S(a_n)$ is closed, let L be any limit point of $S(a_n)$. Then there is a sequence (s_k) such that for each $k \in$ N, $s_k \in S(a_n)$, $s_k \neq L$ and $\lim_{k\to\infty} s_k = L$. Because each s_k is a sublimit of (a_n), there is a subsequence $(a_n(k))$ such that $\lim_{i\to\infty} a_n(k,i) = s_k$. (See Figure 1.) We need to build a new subsequence $(a_n(L))$ converging to L in order to show $S(a_n)$ is closed.

For $i \in$ N, let $j(i)$ be the smallest subscript such that for $|s_{j(i)} - L| < \frac{1}{i}$ and let $a_n(j(i), h(i))$ be the first term of (a_n) such that $|a_n(j(i), h(i)) - s_{j(i)}| < \frac{1}{i}$. That is, we choose the sublimit $s_{j(i)}$ to be close to our ultimate limit L and in turn choose the term $a_n(j(i), h(i))$ to be close to $s_{j(i)}$. Thus $a_n(j(i), h(i))$ must be fairly close to L. More precisely, $|a_n(j(i), h(i)) - L| < \frac{2}{i}$. Note that without uniform convergence of the subsequences $(a_n(k))$ we need the Axiom of Choice to ensure the existence of all of the $h(i)$.

We are now ready to define our subsequence $(a_n(L))$ recursively. We take $a_n(L,1) = (a_n(j(1), h(1)))$. Given $a_n(L,w)$, define $a_n(L, w+1)$ to be the first term $a_n(j(i), h(i))$ such that $i \geq (w+1)$ and $a_n(j(i), h(i))$ has a larger index in the original sequence (a_n) than $a_n(L,w)$ has. Since there are infinitely many terms $a_n(j(i), h(i))$, there are terms satisfying these conditions. Then we have $|a_n(L,w) - L| < \frac{2}{w}$ and the subsequence $a_n(L,w)$ converges to L. Hence L is a sublimit of (a_n) and $S(a_n)$ is closed.

Now that we know the set of sublimits is closed, we turn the situation around and show in Theorem 2 that every closed set of reals is a set of sublimits. The proof of Theorem 2 is more involved than the first proof since it requires constructing a sequence to fit a given closed set and ensuring that no extraneous sublimits sneak in.

Theorem 2. For any closed subset F of R there is a sequence (a_n) such that $S(a_n) = F$.

Proof. We may assume that the closed set F is non-empty since otherwise we could use the sequence of Example 2. To simplify notation, we further assume $0 \in F$. (If $0 \notin F$ but $a \in F$, we adapt the following construction by adding a throughout.)

To approximate every element of F we first define a collection of intervals $I_{i,j}$, where $i \in \mathbb{N} \cup \{0\}$ and $1 \leq j \leq 2 \cdot 4^i$. (See Figure 2.) Define
$I_{i,j} = [\, -2^i + (j-1)(2^{-i}), \ -2^i + (j)(2^{-i})]$ for $i \geq 0$ and $1 \leq j \leq 2 \cdot 4^i$. Note that
$$\bigcup_{j=1}^{2\cdot 4^i} I_{i,j} = [-2^i, 2^i].$$
Thus each time i increases, the family of intervals $\{I_{i,j} : 1 \leq j \leq 2 \cdot 4^i\}$ covers an interval twice as long with intervals half as long.

For each interval $I_{i,j}$ we choose a number $a_{i,j}$, which may or may not be in the interval. If $F \cap I_{i,j}$ is non-empty, we let $a_{i,j}$ be the midpoint of $I_{i,j}$. Otherwise, $a_{i,j} = 0$. (Figure 3 illustrates the numbers $a_{i,j}$ for a specific set F and the intervals $I_{i,j}$.) We use the lexicographic order on the numbers $a_{i,j}$ to obtain a sequence. That is, $a_{i,j}$ comes before $a_{n,k}$ if and only if $i < n$ or ($i = n$ and $j < k$). [The sequence starts off $a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{1,3}, \ldots,$ $a_{1,8}, a_{2,1}$, etc.] Let b_n be the nth term of the $a_{i,j}$ using this ordering.

Claim: F is the set of sublimits of (b_n). First we show that if $x \in F$, then $x \in S(b_n)$, and then we show the converse. Let $x \in F$. There is $n \in \mathbb{N}$ such that $|x| \leq 2^n$. For $i \geq n$, there is one (or possibly two) choices of j such that $x \in I_{i,j}$. For these i and j, we see that $|x - a_{i,j}| \leq 2^{-i-1}$ because $a_{i,j}$ is the midpoint of an interval of length 2^{-i}. Thus we can form a subsequence $(b_n(x))$ from these $a_{i,j}$ and $(b_n(x))$ converges to x. So x is a sublimit of (b_n).

Suppose $y \notin F$. Since F is closed, there is $\epsilon > 0$ such that the interval $(y - \epsilon, y + \epsilon)$ is disjoint from F. However, that doesn't mean that each $a_{i,j}$ must be at least ϵ away from y. If there is $w \in I_{i,j} \cap F$, then $a_{i,j}$, the midpoint of $I_{i,j}$, satisfies $|w - a_{i,j}| \leq 2^{-i-1}$. Since $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$. Hence for $i \geq n$ and any j, the closest $a_{i,j}$ could be to y is $|y - a_{i,j}| \geq |y - w| - |w - a_{i,j}| \geq 2^{-n-1}$. Thus no subsequence of (b_n) can converge to y and $S(b_n) = F$, as claimed.

The proof of Theorem 1 generalizes readily to any metric space. The proof of Theorem 2 generalizes to $\mathbb{R}^n$ by replacing the intervals $I_{i,j}$ with n-dimensional "boxes." However, Example 4 below shows that there are metric spaces for which Theorem 2 fails.

Example 4. Let F be the set of all real functions $f : \mathbb{R} \to [0, 1]$ and define a metric d on F by $d(f, g) = \sup|f(x) - g(x)|$. The whole space is closed, as is any metric space. However, we will show that no sequence of functions has the whole space as its set of sublimits. Let (f_n) be any sequence in F. Consider the new function $f : \mathbb{R} \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{N} \\ f_n(n) + 0.5 & \text{if } x = n \text{ and } f_n(n) \leq 0.5 \\ f_n(n) - 0.5 & \text{if } x = n \text{ and } f_n(n) > 0.5 \end{cases}.$$

Then $d(f, f_n) \geq |f(n) - f_n(n)| = 0.5$. Thus no subsequence of (f_n) can approach the function f.

The reader is invited to determine some of the many other closed subsets of F that are not sets of sublimits.

The key to generalizing Theorem 2 successfully is the existence of a subset like the midpoints of the intervals $I_{i,j}$, which is a countable, dense subset of R. A subset S of a metric space X is *dense* in X if and only if the closure of S is X. Equivalently, S is dense in X if and only if for every $x \in X$ there is a sequence of elements of S converging to x. For a sequence to have the whole space as its set of limit points, the sequence as a set must be dense in the space. Since sequences have countably many terms, only spaces with countable dense subsets can be candidates to generalize Theorem 2. Theorem 3 below assures us they do. The reader is encouraged to prove Theorem 3 assuming the following fact, proven in Kuratowski [2, 156]: If a metric space has a countable dense subset, then every subset of it does too. The reader should also consider why we need to require $F \neq \varnothing$ in this theorem.

Theorem 3. If a metric space X has a countable dense subset and F is a non-empty closed set in X, then there is a sequence (a_n) whose set $S(a_n)$ of sublimits is F.

The close connection of sublimits with the deeper idea of closed sets helps explain why sublimits have not been studied more extensively for their own sake.

References

1. Stephen Abbott, *Understanding Analysis*, Springer, New York, 2001.
2. Kazimierz Kuratowski, *Introduction to Set Theory and Topology*, Addison Wesley, Reading, Mass., 1962.
3. Sining Zheng and Jiangchen Cheng , Density of the images of integers under continuous functions with irrational periods, this Magazine 72 (1999), 402-404.