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Deconstructing Bases
Fair, Fitting, and Fast Bases

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Elementary school students wrestling with decimals quickly realize not all fractions are created equal. While the relatively awkward fraction $\frac{17}{32}$ turns into the modestly nice finite decimal 0.53125, other seemingly simple ones, like $\frac{2}{3} = 0.66666666\ldots$ or $\frac{1}{7} = 0.14285714\ldots$, confront us with the sophisticated notion of infinite repeating decimals. Can we find a "finitely fair" base, one in which all fractions have finite representations?

It is natural to start our hunt for a finitely fair base by changing from the familiar base 10 to base b, where b is any integer greater than 1. Unfortunately, the following review of representations base b reveals that for any b some fractions must have infinite repeating representations, while other fractions have finite representations. Recall that $0.a_1a_2a_3\ldots_b = \frac{a_1}{b} + \frac{a_2}{b^2} + \frac{a_3}{b^3} + \ldots = \sum_{n=1}^{\infty} \frac{a_n}{b^n}$, where the subscripted b indicates the base and a_n is an integer satisfying $0 \le a_n \le b - 1$. A fraction $\frac{p}{q}$ in reduced form has a k-place representation in base b exactly when q divides b^k but doesn't divide b^{k-1}. For example, $32 = 2^5$ divides 10^5 but not 10^4, so base 10 uses five decimal places for $\frac{17}{32}$. If q has a prime factor not in b, the reduced fraction $\frac{p}{q}$ has an infinite repeating representation base b. Since no fixed b has every prime factor, every base has some fractions with infinite repeating representations.

Clearly, a finitely fair base requires something new, a mathematical "deconstruction" of the idea of a base. The postmodern term "deconstruction" describes something mathematicians have done two centuries: probe a familiar concept more deeply to find new interpretations and understandings. An initial deconstruction of base in the next section leads to a finitely fair base, called base $\{n!\}$. A further deconstruction in the middle section leads more generally to variable bases, which we use to find bases that "fit" a given real number with a specified representation. The final section critiques bases in yet another way, leading to competing measures for finitely fair bases. A curious use of the Prime Number Theorem shows that we can control both of these measures. For ease we consider only representations of real numbers between 0 and 1, although the reader is invited to extend these ideas to the integer parts.

A Finitely Fair Base Finite representation for all fractions requires a reinterpretation of the concept of a base. The familiar infinite series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ suggests replacing the powers b^n in the denominators of the base with the factorials $n!$. We will denote this new base with the subscript $_{\{n!\}}$. For example, $\frac{1}{3!}$ becomes $0.001_{\{n!\}}$ and $\frac{3}{5!}$ becomes $0.00003_{\{n!\}}$. Then $e = 2 + \sum_{n=2}^{\infty} \frac{1}{n!}$ has the memorable representation $2 + 0.01111\ldots_{\{n!\}}$ and $\frac{5}{6}$ equals $0.012_{\{n!\}}$.

Definition. By $0.a_1a_2a_3\ldots_{\{n!\}}$ we mean $\sum_{n=1}^{\infty} \frac{a_n}{n!}$, where a_n is an integer satisfying $0 \le a_n \le n - 1$.

The first place in base $\{n!\}$ is always a useless 0, but retaining it makes the n^{th} place correspond to $\frac{1}{n!}$.) Since q divides $q!$, the representation of $\frac{p}{q}$ never needs more than q places, showing base $\{n!\}$ is finitely fair. Indeed, only when q is a prime or 4 does the reduced fraction $\frac{p}{q}$ need q places.

The condition $0 \le a_n \le n - 1$, which corresponds to the condition $0 \le a_i \le b - 1$ in base b, ensures that every fraction has a unique finite representation. Suppose, to illustrate the uniqueness of this representation, we try to find a second representation for $\frac{1}{3!} = 0.001_{\{n!\}}$. The most we can put in the fourth place is a 3, but $0.0003_{\{n!\}} = \frac{3}{4!}$, which is $\frac{1}{3!} - \frac{1}{4!}$ since $\frac{1}{3!} = \frac{4}{4!}$. Again, the most we can add in the fifth place is a 4, and $0.00034_{\{n!\}} = \frac{1}{3!} - \frac{1}{5!}$. In general $\frac{i-1}{i!} = \frac{i}{i!} - \frac{1}{i!} = \frac{1}{(i-1)!} - \frac{1}{i!}$, and so $0.000345\ldots(k-1)_{\{n!\}} = \frac{1}{3!} - \frac{1}{k!}$.

Irrationals related to e can have nice representations base $\{n!\}$. Recall the series $\cosh(x) = \frac{e^x+e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$. Then $\cosh(1) = 1 + 0.010101\ldots_{\{n!\}}$, and similarly $\sinh(1) = 1 + 0.00101010\ldots_{\{n!\}}$. Since $e^{-1} = \cosh(1) - \sinh(1)$ we can use elementary borrowing in base $\{n!\}$ to find the representation of e^{-1}:

$$\begin{array}{rl} & 1.010101010\ldots_{\{n!\}} \\ - & 1.001010101\ldots_{\{n!\}} \\ = & 0.002040608\ldots_{\{n!\}} \end{array}$$

In base $\{n!\}$ the representation of π starts off $3 + 0.00031565\ldots_{\{n!\}}$. It seems unlikely that the integers in this representation will form a pattern anyone could describe, although that might well be a deep mathematical question. Some might hope to improve on base $\{n!\}$ so that every irrational would have a repeating representation or other easily recognized pattern. However a cardinality argument shows that not all of the uncountably many real numbers can have recognizable patterns, even in such a lovely base as $\{n!\}$. Whatever the term "recognizable" might mean, such a pattern must at least be describable in finitely many terms. For example we can describe the pattern $0.00204060\ldots_{\{n!\}}$ for e^{-1} by "the digit in the $2n+1^{st}$ place for $n \geq 1$ is $2n$ and elsewhere is 0." In any language or base there are only countably many symbols and so only countably many things describable in a finite number of terms. (See [Fletcher and Patty, 223].) Thus no matter what base we invent, there will always be uncountably many real numbers with nonrepeating, and even indescribable representations. However in the next section we see how to find a base fitting any real number with almost any desired representation.

Finding Fitting Bases. For any given real number between 0 and 1 we seek a base with a specified representation of that number. First we need to deconstruct the idea of a base beyond base $\{n!\}$ or base b. Note that the successive denominators in either case $(1, 2, 6, 24, \ldots$ or $b, b^2, b^3, \ldots)$ are multiples of previous denominators. To see the advantage of this property, consider an attempted base $\{\$\}$ built on making change with American coins. We could write $0.a_1a_2a_3a_{4\{\$\}} = \frac{a_1}{4} + \frac{a_2}{10} + \frac{a_3}{20} + \frac{a_4}{100}$, where a_1, a_2, a_3 and a_4 are the number of quarters, dimes, nickels and pennies, respectively. To make change, we never need more than four pennies, so we restrict $0 \leq a_4 \leq 4$. Similarly, $0 \leq a_3 \leq 1$ since two nickels equals a dime. We need up to two dimes to make change, so $0 \leq a_2 \leq 2$. Even with this restriction, we don't have a unique way to make change since $0.0210_{\{\$\}} = 0.1000_{\{\$\}}$. This ambiguity occurs because a quarter can't be evenly divided into dimes.

Definition. A sequence $\{b_n\}$ of positive integers is a *variable base* provided each b_n divides b_{n+1} and $\lim_{n\to\infty} b_n = \infty$. Define *place ratios* r_n recursively by $r_1 = b_1$ and $r_{n+1} = \frac{b_{n+1}}{b_n}$. We define the base $\{b_n\}$ *representation* $0.a_1a_2a_3\ldots_{\{b_n\}}$ to equal $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$, where a_n is an integer satisfying $0 \leq a_n \leq r_n - 1$.

Base b and base $\{n!\}$ clearly qualify as variable bases: in base b, we have $b_n = b^n$ and $r_n = b$, whereas $b_n = n!$ and $r_n = n$ in base $\{n!\}$. Recall the earlier example showing that $\frac{1}{3!}$ has a unique finite representation in base $\{n!\}$. The keys underlying that argument generalize to the conditions $b_n = r_nb_{n-1}$ and $0 \leq a_n \leq r_n - 1$, which correspond to the requirements for a variable base representation Thus for all variable bases $\{b_n\}$ finite representations in base $\{b_n\}$ are unique. Because $\lim_{n\to\infty} b_n = \infty$ every real between 0 and 1 is approximated by finite representations, and so it has a finite or infinite representation base $\{b_n\}$.

Let's take as our test case π, whose representation in base $\{n!\}$ appears as indescribable as its representation in base 10. Can we find some $\{b_n\}$ so that $\pi = 3 + 0.1111\ldots_{\{b_n\}}$? For a_1 to be 1, $\frac{1}{b_1} < 0.14159\ldots_{10} < \frac{2}{b_1}$. Then $8 \le b_1 \le 14$. Once we pick b_1, say $b_1 = 8$, we have $\frac{1}{b_2} < 0.14159\ldots_{10} - \frac{1}{b_1} = 0.01659\ldots_{10} < \frac{2}{b_2}$. Thus $60 < b_2 < 120$. Further, b_2 must be a multiple of $b_1 = 8$. If we always pick the smallest denominator at each step, we get the base $\{b_n\} = \{8, 8^2, 8^2 \cdot 17, 8^2 \cdot 17 \cdot 19, \ldots\}$, although this sequence seems no more memorable than the digits of π in base 10. Still, Theorem 1 below provides an explicit construction of a base fitting any number between 0 and 1 with a specified infinite representation.

Theorem 1. Let $0 < \gamma < 1$ and let $\{a_n\}$ be any sequence of positive integers. Then there is a base $\{b_n\}$ such that $\gamma = 0.a_1a_2a_3\ldots_{\{b_n\}} = \sum_{n=1}^{\infty} \frac{a_n}{b_n}$.

Proof. We use recursion to construct a base $\{b_n\}$. Because $0 < \gamma$ and $\lim_{k\to\infty} \frac{a_1}{k} = 0$, there are values k such that $\frac{a_1}{k} < \gamma$. Let b_1 be the smallest k such that $\frac{a_1}{k} < \gamma < \frac{a_1+1}{k}$. Set $\gamma_1 = \gamma - \frac{a_1}{b_1}$, so that $0 < \gamma_1 < \frac{1}{b_1} = \frac{a_2}{a_2 b_1}$. Choose b_2 to be the least multiple of b_1 such that $\frac{a_2}{b_2} < \gamma_1 < \frac{a_2+1}{b_2}$. In general, suppose we have b_i for $1 \le i \le n$ such that b_i divides b_{i+1} for $i < n$ and $\sum_{i=1}^{n} \frac{a_i}{b_i} < \gamma < \left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) + \frac{1}{b_n}$. For $\gamma_n = \gamma - \sum_{i=1}^{n} \frac{a_i}{b_i}$ we have $0 < \gamma_n < \frac{1}{b_n} = \frac{a_{n+1}}{b_n a_{n+1}}$. Let b_{n+1} be the least multiple of b_n such that $\frac{a_{n+1}}{b_{n+1}} < \gamma_n < \frac{a_{n+1}+1}{b_{n+1}}$. Clearly b_n properly divides b_{n+1} so $\lim_{n\to\infty} b_n = \infty$ and $\{b_n\}$ is a base. Further, $\sum_{i=1}^{n+1} \frac{a_i}{b_i} < \gamma < \left(\sum_{i=1}^{n+1} \frac{a_i}{b_i}\right) + \frac{1}{b_{n+1}}$ and $\lim_{n\to\infty} \frac{1}{b_{n+1}} = 0$. Hence $\sum_{i=1}^{\infty} \frac{a_i}{b_i} = \gamma$.

While the preceding proof chooses the smallest possible b_{n+1} each time, there may be other choices of bases fitting the same representation to γ. For example by suitably continuing the sequence $\{c_n\} = \{9, 36, 396, 5940, \ldots\}$ we would also have $\pi = 3 + 0.111\ldots_{\{c_n\}}$. Determining how many different bases fit a given real with a specific representation appears quite difficult. We illustrate the range of possibilities for the number of bases giving a particular representation using perhaps the easiest examples. The reader is invited to fill in the details of the induction proofs.

–There is a unique base $\{b_n\}$ (namely $b_n = 3^n$) such that $\frac{1}{2} = 0.111\ldots_{\{b_n\}}$.

–There are uncountably many bases $\{b_n\}$ such that $\frac{1}{3} = 0.111\ldots_{\{b_n\}}$.

Sketch of the proof. For each $k \in \mathbb{N}$ consider the options ($r_{2k-1} = 4$ and $r_{2k} = 4$) and ($r_{2k-1} = 5$ and $r_{2k} = 2$). Using the notation in the preceding proof, either way $\gamma_{2k} = \frac{1}{3b_{2k}}$, so the next step will have the same options. Since $2^{\mathbb{N}}$ is uncountable, there are uncountably many ways of choosing the values r_n in pairs. (There are other possible bases as well.)

–There is a countable infinity of bases $\{b_n\}$ such that $\frac{1}{4} = 0.111\ldots_{\{b_n\}}$.

Sketch of the proof. If the first k choices of r_n are all 5 then $\gamma_k = \frac{1}{4b_k}$ and we have three choices for r_{k+1}, namely 5, 6, and 7. Choosing $r_{k+1} = 5$ leaves these options open for r_{k+2}. However, if $r_{k+1} = 6$, then $r_{k+i} = 3$ for all $i \ge 2$. Similarly, the choice of $r_{k+1} = 7$ forces $r_{k+2} = 2$ and for all $i \ge 3$, $r_{k+i} = 3$. Thus there are countably many times one can choose to deviate from $r_n = 5$. However, once that choice is made, there are no further options.

We abandon the difficulties of fitting bases to desired representations of given numbers. Instead, we return our attention to finitely fair bases, considering competing comparisons of them with base $\{n!\}$.

Fast Finitely Fair Bases. To consider alternatives to base $\{n!\}$ we need to determine when a base is finitely fair. Minimally every fraction $\frac{m}{p^k}$, where p is a prime, needs to be finitely represented. That is, for all primes p and all natural numbers k, there is a natural number n such that p^k divides b_n. Actually this condition also suffices: factor the denominator of $\frac{m}{q}$ into powers of primes $q = p_1^{k_1} \dots p_j^{k_j}$ and pick b_n to be the maximum of the b_i corresponding to the $p_i^{k_i}$. While base $\{n!\}$ is finitely fair, it seems slow in one way and fast in another. First of all it is "slow" in that it can need as many as q places to represent $\frac{p}{q}$. However, its place ratios, $r_n = n$, grow fast and without bound. Each criterion alone seems mathematically uninteresting: we could "speed up" how quickly we represent fractions by choosing huge values for the r_n, or we could, on average, slow the growth of the place ratios small by taking "almost all" of the r_n to be 1, sprinkling in the primes just often enough to get infinitely many of each eventually. Combining these two senses of speed leads to a more interesting result. We measure the overall growth of the place ratios r_n with their geometric mean rather than their arithmetic mean because of their multiplicative nature.

Definition. The n^{th} *average place ratio* of $\{b_n\}$ is $\sqrt[n]{b_n} = \sqrt[n]{\prod_{i=1}^{n} r_i}$.

For base b the n^{th} average place ratio is constant: $\sqrt[n]{b^n} = b$. From Stirling's approximation for $\{n!\}$ the n^{th} average place ratio for base $\{n!\}$ is $\sqrt[n]{n!} \approx \sqrt[n]{\sqrt{2\pi n}\,(\frac{n}{e})^n} > \frac{n}{e}$, which goes to infinity as n does. (See [Woodroofe, 127-128].)

Definition. A finitely fair base $\{b_n\}$ is *q-fast* if the representation in base $\{b_n\}$ of a reduced fraction $\frac{p}{q}$ never needs more than q places.

Although base $\{n!\}$ is q-fast, factorials lead to fairly fast growth of the average place ratios. Fortunately, a q-fast base doesn't need factorials. We need $r_2 = 2$, and $r_3 = 3$ to handle halves and thirds by the second and third place, respectively. But we only need $r_4 = 2$ to accommodate fourths in the fourth place since $r_4 = 2$ gives $b_4 = 2 \cdot 3 \cdot 2 = 12$, a multiple of 4. Similarly, $r_6 = 1$ suffices since $b_4 = 12$ is already a multiple of 6. The *slowest q-fast base* increases the denominators b_n only as much as needed. This base has the sequence of smallest place ratios, which are $r_{p^k} = p$ for every power of a prime p^k and $r_n = 1$ otherwise. The following table gives initial values of r_n, b_n and $\sqrt[n]{b_n}$ for this slowest q-fast base.

n	1	2	3	4	5	6	7	8	9	10	11
r_n	1	2	3	2	5	1	7	2	3	1	11
b_n	1	2	6	12	60	60	420	840	2520	2520	27720
$\sqrt[n]{b_n}$	1	1.41	1.86	1.9	2.27	1.98	2.37	2.32	2.39	2.19	2.53

Table 1. Values of r_n, b_n and $\sqrt[n]{b_n}$ for the slowest q-fast base.

While the preceding table suggests the average place ratios $\sqrt[n]{b_n}$ increase slowly but unevenly, surprisingly the sequence $\{\sqrt[n]{b_n}\}$ has a bound related to e.

Theorem 2. The slowest q-fast base $\{b_n\}$ has $\sqrt[n]{b_n} < e^{1.105} \approx 3.02$ for all n.

Proof. In a q-fast base if $n \geq p^k$, for a power of a prime, then p^k must divide b_n. The slowest q-fast base has no extra powers of any p, meaning when we factor b_n into primes, for any prime p there are exactly k factors of p in b_n, where $p^k \leq n < p^{k+1}$. The needed power of a prime p is $\lfloor \log_p n \rfloor$, where $\lfloor x \rfloor$ is the floor function of x, the greatest integer less than or equal to x. Hence $b_n = \prod_{p=2}^{n} p^{\lfloor \log_p n \rfloor}$, where p varies over all primes between 2 and n. Now $\prod_{p=2}^{n} p^{\lfloor \log_p n \rfloor} < \prod_{p=2}^{n} p^{\log_p n}$ and $p^{\log_p n} = n$, so $b_n < n^{p(n)}$, where $p(n)$ is the number of primes less than or equal to n. The Prime Number Theorem gives the approximation $p(n) \approx \frac{n}{\ln(n)}$ with an upper bound of $p(n) < \frac{1.105n}{\ln(n)}$. (See [Kline, 830].) Thus $\sqrt[n]{b_n} < (n^{1.105n/\ln(n)})^{1/n} = n^{1.105/\ln(n)} = n^{1.105 \log_n e} = e^{1.105}$.

Remarks. The value $\sqrt[n]{b_n}$ exceeds e for the first time when $n = 19$, giving approximately 2.76.

The maximum value of $\sqrt[n]{b_n}$ appears to be just under 2.8 when $n = 31$.

We can readily generalize the idea of the slowest q-fast base. A base $\{b_n\}$ is *βq-fast* if its representation of $\frac{p}{q}$ never needs more than βq places (or 1 place if $\beta q < 1$). For example, the slowest $\frac{q}{2}$-fast base needs $r_1 = b_1 = 6$ to take care of halves and thirds by the first place, $r_2 = 10$ to take care of fourths and fifths by the second place, and so on. The preceding theorem generalizes as follows.

Corollary. The slowest βq-fast base $\{b_n\}$ has $\sqrt[n]{b_n} < e^{1.1/\beta}$.

Proof. We need only replace $b_n < n^{p(n)}$ in the previous proof with $b_n < n^{p(n)/\beta}$.

Thus a suitable fast finitely fair base can slow the growth of the place ratios and simultaneously represent fractions as quickly as desired.

Mathematicians derive much pleasure from “deconstructing” a commonly accepted concept to find a more fundamental and general one. Often these generalizations give profound insights and applications. Others, such as fair, fitting, and fast bases, are simply fun.

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