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Nim on Groups

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Nim on Groups

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Abstract

The traditional game of Nim comprises of two players who take turns removing objects from distinct piles. The player who takes the last object is the winner. We consider the game Nim on Cayley graphs of finite groups, where the piles are located on the vertices and the number of objects in each pile is denoted as the *weight* of the vertex. In this version of the game, a player wins by trapping the opponent on a vertex with weight zero so he or she is unable to further reduce the weight of that vertex. We examine winning strategies for Nim on Cayley graphs of cyclic groups, dihedral groups, and the Quaternions, among others.

Chapter 1

Introduction

1.1 Introduction of Nim

Nim is a simple mathematical game that has existed for years. In fact, many believe it originated in China and is one of the oldest games in the world [8]. The name Nim is attributed to Charles Bouton and is said to be related to the German term *zu nehmen*, or “to take” [8]. Nim has been played using three distint piles of anything from stones, matchsticks, tokens, or any other counter [2]. Each pile contains some amount of stones, but there may be different numbers of stones among the piles. In its basic form, two players compete to be the one to take the last stone of the game. Players alternate turns until there is a winner. On each turn, a player chooses a single pile to remove stones. The player may remove as many stones as desired, as long as it is at least one, from the single pile chosen. Then the other player takes a turn. Nim continues in this fashion until a player takes the last stone. Then that player wins because there are no more stones to remove.

According to Bouton [2], the winning strategy for the traditional game of Nim is based on the set of numbers a player leaves at the end of the turn. If a player leaves a certain set of numbers on the table and plays without mistake, then the other person cannot win. Bouton designates these winning strategies as safe combinations. He shows if a player leaves a safe combination at the end of a turn, then the other person cannot leave a safe combination. Thus the player will always take the last counter and win. This is one example of how winning strategies of Nim are determined. The strategies employed for ordinary Nim, however, are not applicable to Nim on groups.

The game of Nim can be quite flexible. Simple variations include altering the number of piles and the number of tokens in the piles. Other variations on the structure of Nim games have been proposed and studied. Previous research involving Nim on graphs was introduced in papers by Fukuyama [5] and [6]. The game was also extended by Erickson [3]. In their variation of Nim, arbitrary weights are assigned to the edges of different graphs, including bipartite and multiple edge graphs among others. The object of Nim on graphs is to trap the other player on a vertex in which all adjacent edges have a weight of zero. Fukuyama uses Grundy numbers to generate winning strategies, which are based on certain positions of the graph. Erikson provides results for Nim on paths, cycles, and complete graphs.

I follow a similar procedure in constructing the game Nim on groups. In my variation, the gameboard is the directed Cayley graph of a group. Instead of placing the weights on the edges, I place the weights on the vertices of the graph. Each Nim game we examine will differ in the structure of the gameboard. The winning strategies presented account for any possible weight distributions. In fact, we will see the weight distribution is often the determining factor in gameplay. The goal of this research is to yield interesting group properties based on the behavior of Nim on Cayley graphs.

1.2 Basics of Groups

The following section is a brief review of basic concepts of group theory the reader should be familiar with before proceeding. It contains formal definitions necessary in building the gameboards on which we play Nim. We start with defining a *group* [9].

Definition 1.1. *A group G is any set with an operation (denote by juxtoposition) that satisfies the following four axioms:*

1. *There is an identity element $e \in G$ such that $eg = g = ge$ for all $g \in G$.*
2. *For every $g \in G$, there is an element $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$.*
3. *For every $g, h, k \in G$, $(gh)k = g(hk)$.*
4. *For every $g, h \in G$, $(gh) \in G$.*

The *order* of a group G, denoted $|G|$, is the number of elements in the set G. A group G is *abelian* if $gh = hg$ for all $g, h \in G$, and *non-abelian* if there is some $g \in G$ and some $h \in G$ such that $gh \neq hg$. We will be working with the Cayley graphs of both abelian and non-abelian groups. Here we take some time to define what it means to be a *Cayley graph*.

1.3 Constructing Cayley Graphs

Since the entirety of this paper relies on Cayley graphs, it is important to grasp exactly how we construct these gameboards for Nim. We first define a *Cayley graph* as seen introduced in [4].

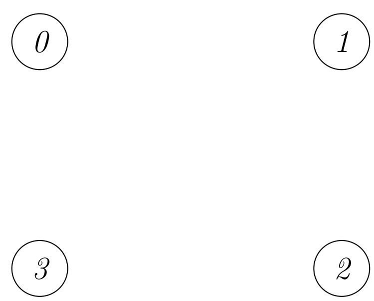
Definition 1.2. *Let G be a finite group and let $S \subseteq G$ be a subset. The corresponding Cayley graph $C(G, S)$ has a vertex set equal to G. Two vertices $a, b \in C(G, S)$ are joined by a directed edge from a to b if and only if there exists $s \in S$ such that $b = as$. We then call S the generating set with all $s \in S$ generators of G.*

Note the Cayley graph will always contain a representation of every element in the corresponding group. Then there are $|G|$ vertices in the Cayley graph of G. We will only consider *finite* groups, which contain a finite number of elements.

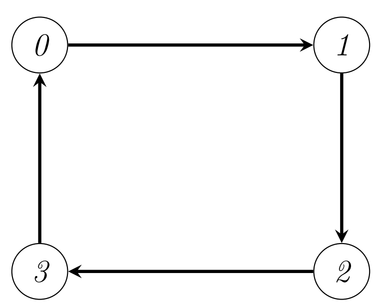
We first designate a group G on which to play the game Nim. To construct the corresponding Cayley graph, we start with assigning each $x \in G$ to a vertex. Then there are $|G|$ vertices, with each vertex labeled as its corresponding element name in G. Next we must choose a generating set. Let $S = \{g_1, g_2, ..., g_k\}$ be a generating set for G. There are k edges directed from every vertex x to xg_i for $1 \leq i \leq k$. These edges determine the players' movements in the game. For example, if there exists a g_i such that $b = ag_i$ for any vertex a, b in the Cayley graph, then the player may move from a to b. The edge would be a directed arrow pointing from a to b. The player may not move from b to a, however, unless there exists some $g_j \in S$ such that $a = bg_j$. If this is the case, then the edge would be a double sided arrow connecting a and b.

The following is an example of how we construct the Cayley graph of a *cyclic group* with a single generator $S = \{1\}$. Chapter 2 defines cyclic groups explicitly.

Example 1. *For the Cayley graph of Z_4, we first draw and label a vertex corresponding to each element of Z_4.*



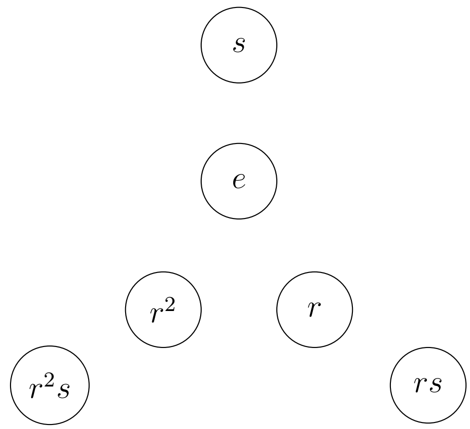
Then we draw the edges using the generator 1. *We draw an arrow from each vertex x to $x + 1$.*



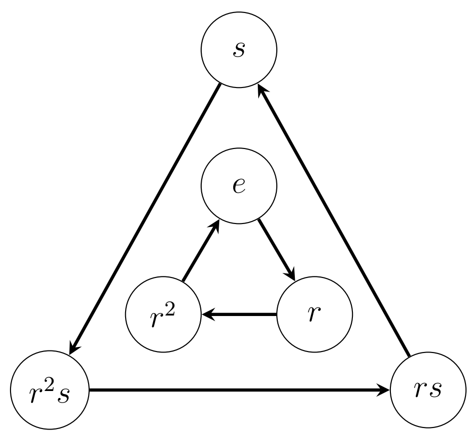
This gives us the Cayley graph of Z_4.

The next example shows how we construct the Cayley graph of a group with multiple generators. Consider the Cayley graph of a *dihedral group* with the generating set $S = \{r, s\}$. Chapter 3 defines dihedral groups explicitly.

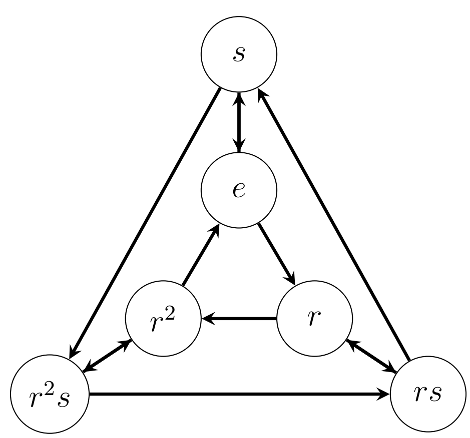
Example 2. *For the Cayley graph of D_3, we first draw and label a vertex corresponding to each element of D_3.*



Now we consider the generator r. We draw an arrow from each vertex x to xr.



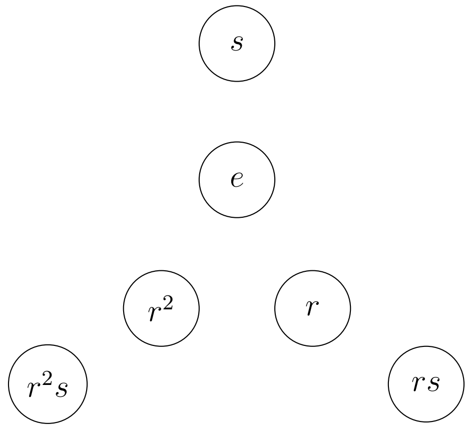
Finally we consider the generator s. We draw an arrow from each vertex x to xs. Since $(xs)s = x$, the arrows between x and xs will be double-sided arrows.



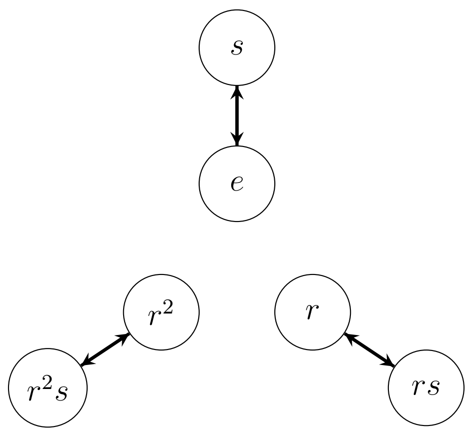
This gives us the Cayley graph for D_3.

Finally we will construct the Cayley graph of D_3 using different generators. Consider the generating set $S = \{rs, s\}$.

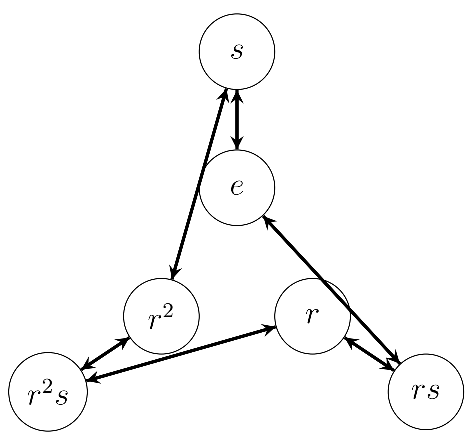
Example 3. *Again we first draw and label vertices that correspond with the elements of D_3.*



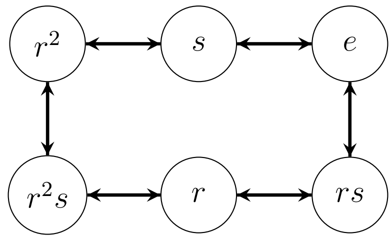
Now we consider the generator s. We draw an arrow from each vertex x to xs.



Now we consider the generator rs. We draw an arrow from each vertex x to xrs.



We can rearrange these vertices to obtain the Cayley graph of D_3 with generators s and rs.



These examples illustrate the Cayley graphs of different groups as well as different generating sets within the same group.

1.4 Basic Rules of the Game

We define the *weight* of a vertex similarly to the weight of an edge found in [3].

Definition 1.3. *Given a graph G with a vertex set $V(G)$, the non-negative integer value assigned to each $v \in V(G)$ is the weight of the vertex. We denote w_v as the weight of v.*

To begin a game of Nim, we first construct the Cayley graph G of the group of choice as outlined in the previous section. We will associate G with the vertices and edges of $C(G, S)$ and use the names accordingly. Then we assign each vertex $v \in G$ an initial weight, denoted $w_{v,0}$. The vertex weight must be assigned such that $w_{v,0} > 0$. Note that $w_{v,0}$ is the initial vertex weight, while w_v is the current weight at any other point in the game.

Nim proceeds according to a general set of rules.

1. Player 1 begins at the identity vertex e.
2. Player 1 reduces w_e by some amount r such that $1 \leq r \leq w_e$.
3. Player 1 moves to some $eg_i \in G$, where $g_i \in S$.
4. Player 2 reduces w_{eg_i} by some amount s such that $1 \leq s \leq w_{eg_i}$.
5. Player 2 moves to some $eg_ig_j \in G$, where $g_j \in S$.
6. Players continue to alternate turns until there is a winner.

A player wins if his or her opponent starts a turn at vertex m with $w_m = 0$. The opponent cannot remove any more weight since the weight of any vertex can never be less than 0. Then the opponent is said to be trapped, and the player wins.

Note that Nim on groups differs from the general game in that it might not be the case that all weight is removed from the graph before the game is over. All it takes to win in this variation is for a player to move to a single vertex with weight 0. We will use the *Player* and the *Opponent* to differentiate between players in a general sense. We use these terms consistently when describing strategic moves that can be made by either player. Note that the winning strategies for Nim on each group will always be presented in terms of Player 1 and Player 2. In developing the proofs of these strategies, we assume that a player with a winning strategy will choose to use it, and a player will avoid moving to a vertex which gives the opponent a winning strategy.

Chapter 2

Cyclic Groups

2.1 Introduction

We first look at the game of Nim on cyclic groups.

Definition 2.1. *Let G be a group and let $a \in G$. Then the subgroup*

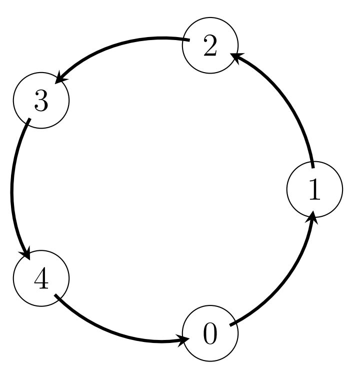
$$\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$$

is called the cyclic subgroup of G generated by a.

Every cyclic group of order n is isomorphic to $\mathbb{Z}_n$, so we can consider every cyclic group of the form $\mathbb{Z}_n = \langle 1 \rangle$ with the binary operation of addition modulo n. The generating set is given by $S = \{1\}$ and $|\mathbb{Z}_n| = n$.

To begin Nim on cyclic groups, we must construct the gameboard. In the directed Cayley graph of $\mathbb{Z}_n$, vertices are labeled $\{0, 1, 2, ..., n-1\}$ with 0 as the identity vertex. Define a vertex as *even* if it is labeled as $2i$ for some $i \in \mathbb{Z}$ and *odd* if it is labeled as $2i + 1$ for some $i \in \mathbb{Z}$. The following example displays the gameboard for Nim on $\mathbb{Z}_5$.

Example 4. *This is the Cayley graph for the cyclic group $\mathbb{Z}_5$ when $S = \{1\}$.*



2.2 Nim on $\mathbb{Z}_n$

We are able to determine the winning strategy for Nim on cyclic groups assuming each vertex has a positive initial weight and the game proceeds according to the basic rules of Nim. We start with the odd cyclic groups.

Theorem 2.1. *Let G be the Cayley graph for the cyclic group $\mathbb{Z}_n$. If n is odd, then Player 1 has winning strategy.*

Proof. Suppose n is odd. Then $n = 2m+1$ for some $0 \leq m \in \mathbb{Z}$. Player 1 will start the game by setting $w_0 = 0$ and proceeding to 1. Note Player 1 will always reduce w_{2k} and Player 2 will reduce w_{2k+1} for $k \leq m$. Since $w_{r,0} > 0$ for all $r \in G$, no player can lose until a player returns to the identity vertex. Then Player 2 must reduce $w_{2m+1} = w_n = w_0$. Since $w_0 = 0$, Player 1 wins. □

We conclude Player 1 will always have the winning strategy for odd cyclic groups. We now turn to Nim on the Cayley graph of even cyclic groups.

Theorem 2.2. *Let G be the Cayley graph for the cyclic group $\mathbb{Z}_n$, where n is even and $w_0, w_1, ..., w_{n-1}$ are the weights of G where w is the minimum weight of all vertices $x \in G$. Let $m \in \{0, 1, ..., n-1\}$ be the least such that $w_m = w$. Then if m is even, Player 2 has the winning strategy. If m is odd, Player 1 has the winning strategy.*

Proof. Suppose m is even. We will induct on $w_{m,0}$. For the initial case, suppose $w_{m,0} = 1$. Then $w_{x,0} \geq 2$ for $x \in \{0, 1, ..., m-1\}$ if $m > 0$. If $m = 0$, then Player 1 will reduce $w_{m,0}$ to 0. Otherwise, Player 1 will start the game by reducing $w_{e,0}$ by some amount and moving to 1. Player 2's strategy is to reduce the weight of each odd vertex by 1 and proceed to the next vertex. Since m is even and $w_{m,0} = 1$, Player 1 is forced to set $w_{m,0}$ to 0. Play will continue until a player returns to e because $w_{i,0} > 0$ for all $0 \leq i \leq n-1$. Since n is even, Player 1 will necessarily reduce w_e again. Since Player 2 reduced the odd vertices by 1 and $w_{x,0} \geq 2$, Player 2 will not lose on x for $1 \leq x \leq m-1$. If Player 1 reduced $w_{x,0}$ to 0 for some $x \in \{0, 1, ..., m-1\}$, then Player 1 loses at x. Otherwise, $w_m = 0$ and Player 1 is unable to further reduce w_m. Therefore Player 1 loses when $w_{m,0} = 1$.

For the induction step, suppose Player 2 has the winning strategy for $1 \leq w_{m,0} \leq k$. Consider $w_{m,0} = k+1$. Then $w_{x,0} \geq k+2$ for $x \in \{0, 1, ..., m-1\}$. Player 2's strategy is to reduce the weight of each odd vertex by 1 and proceed to the next vertex. Since m is even, Player 1 will be forced to reduce $w_{m,0}$ to $w_m \leq (k+1) - 1 = k$. Play continues until a player returns to e; since n is even, this player is necessarily Player 1. This can be considered a new game on $\mathbb{Z}_n$. Since $w_m \leq k$, the minimal weight of this new game is at most k. Then by the induction hypothesis, Player 2 has the winning strategy.

Suppose m is odd. Then Player 1 has the winning strategy by similar reasoning. □

Together the above theorems yield winning strategies for every possible weight distribution. Thus we are able to say the game Nim on the Cayley graph of cyclic groups is completely solved.

Chapter 3

Dihedral groups

3.1 Introduction

We now turn to dihedral groups, which maintain the form of cyclic groups to some extent and add additional structural components. We start with the definition of a *dihedral group* as seen in [9].

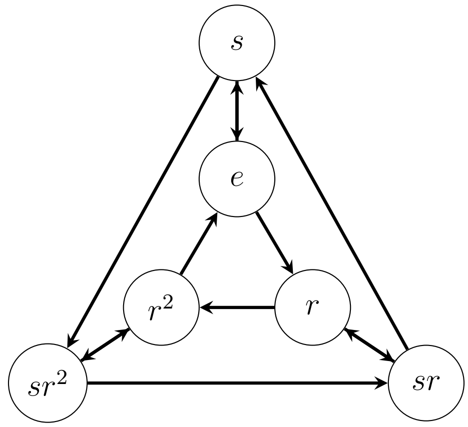
Definition 3.1. *Let $n \geq 3$. We define the nth dihedral group D_n to be the group of rigid motions of a regular n-gon. The group D_n has order $2n$, and the elements consist of all products of r, s satisfying the relations*

$$r^n = e \qquad s^2 = e \qquad srs = r^{-1}$$

Notice r refers to a rotation and s refers to a reflection when viewing D_n in terms of rigid motions. We first construct the gameboard for Nim on dihedral groups using the generating set $S = \{r, s\}$. The Cayley graph of D_n has $|D_n| = 2n$ vertices. Each element in D_n can be written in the form $r^i s^j$ where $0 \leq r \leq n-1$ and $j \in \{0, 1\}$. Thus we label each vertex appropriately. We define a vertex to be *odd* if $i+j$ is odd and *even* if $i+j$ is even.

The edges are directed from any vertex $x \in G$ to xg, where $g \in S = \{r, s\}$. Then each vertex has 2 initiating edges, which means G has $4n$ edges. We call the cycle formed by r^i the *Rotation Cycle*. Notice we can write $r^i s = sr^{n-i}$ by dihedral group properties. Then the cycle formed by $r^i s$ is the *Reflection Cycle*. Each cycle is graph isomorphic to $\mathbb{Z}_n$ and contains no overlap since r^i is never equal to $r^j s$ for any $i, j \in \{0, 1, ..., n-1\}$. These cycles account for $2n$ of the edges. The other $2n$ edges connect r^i to $r^i s$. Since $(r^i)s = r^i s$ and $(r^i s)s = r^i(ss) = r^i$, these are double sided arrows. Finally, when a vertex from the Rotation Cycle is multiplied by r we have $(r^i)r = r^{i+1}$ and the exponent of r increases. When a vertex from the Reflection Cycle, however, is multiplied by r we have $(r^i s)r = (sr^{n-i})r = s(r^{n-i}r) = sr^{n-i+1} = r^{n-(n-i+1)}s = r^{i-1}s$ and the exponent of r decreases. Thus we have shown the Rotation Cycle and the Reflection Cycle have opposite orientations.

Example 5. *This is the Cayley graph of D_3 with $S = \{r, s\}$.*



We say a player *switches cycles* if the player moves from r^i to r^is or from r^is to r^i. Switching cycles is a move that will be important in developing winning strategies for Nim on dihedral groups.

3.2 Techniques

We first examine the general game play techniques for Nim on dihedral groups that we will use to create winning strategies. Since these moves apply to both Player 1 and Player 2, we will consistently refer to one player as the *Player* and the other as the *Opponent*. The following lemmas exemplify such game play techniques.

Lemma 3.1. *A Player is guaranteed victory if the Opponent sets a vertex weight w_x to* 0 *and moves to xs such that $w_{xs} \neq 0$.*

Proof. Let the Opponent begin the turn on some vertex x where $w_x \neq 0$ and $w_{xs} \neq 0$. Suppose the Opponent sets w_x to 0 and moves to xs on the opposite cycle. Since $w_{xs} \neq 0$, the Player reduces w_{xs} by 1 and moves to $xss = x$. Since $w_x = 0$, the Opponent is unable to reduce w_x and the Player wins. □

Therefore, a player should avoid switching cycles after setting the weight of a vertex to 0 to avoid defeat unless the weight of the vertex in the opposite cycle happens to already be 0. In this case, the player would move to xs to win the game and it would not matter how the player reduced w_x.

Lemma 3.2. *Let a Player move to r^i such that $w_{r^i} \leq w_{r^is}$. Either the Player wins or the Opponent moves to r^{i+1}. Similarly, the Player can force the Opponent to move from r^is to $r^{i-1}s$ whenever $w_{r^is} \leq w_{r^i}$.*

Proof. Let the Player move to r^i where $w_{r^i} \leq w_{r^is}$. We will induct on w_{r^i}. For the initial case, suppose $w_{r^i} = 1$. The Opponent reduces $w_{r^i,0}$ to 0 and moves to either r^is or r^{i+1}. Suppose the Opponent does not move to r^{i+1}. Then the Opponent moves to r^is, and the Player wins by Lemma 3.1 since $1 = w_{r^i,0} \leq w_{r^is,0}$.

For the induction step, assume the Player wins or the Opponent moves to r^{i+1} for $1 \leq w_{r^i} \leq k$. Consider $w_{r^i} = k+1$. The Opponent reduces $w_{r^i,0}$ by some amount and moves to either $r^i s$ or r^{i+1}. Suppose the Opponent does not move to r^{i+1}. Then the Opponent moves to $r^i s$. The Player reduces $w_{r^i s,0}$ by 1 and moves to w_{r^i}. Now the Opponent is at r^i. Since $w_{r^i} \leq w_{r^i,0} - 1 = (k+1) - 1 = k$, either the Player wins or the Opponent moves to r^{i+1} by the induction hypothesis.

Suppose the Player moves to $r^i s$ where $w_{r^i s} \leq w_{r^i}$. Then either the Player wins or the Opponent moves to $r^{i-1} s$ by similar reasoning. $\square$

We refer to the situation in the above theorem as the *Forced Play Strategy.* The Player can use the Forced Play Strategy whenever the Opponent tries to switch cycles by moving from a vertex with a lesser weight to a vertex with a greater weight. This strategy will be referred to frequently when defining winning strategies in the next section.

The next strategic technique requires the following foundational definitions. Note the subscript "R" is for "rotation" and "F" is for "reflection" or "flip."

Definition 3.2. *We define the Rotation Triumph Set as*

$$L_R = \{r^i : w_{r^i} \leq w_{r^i s}; w_{r^{i+1}} > w_{r^{i+1}s}\} \qquad 0 \leq i \leq n-1,$$

and we define the Reflection Triumph Set as

$$L_F = \{r^i s : w_{r^i s} \leq w_{r^i}; w_{r^{i-1}s} > w_{r^{i-1}}\} \qquad 1 \leq i \leq n.$$

We use these sets as a way to pick out a *Rotation Triumph Vertex* and a *Reflection Triumph Vertex* that meet the conditions below.

Definition 3.3. *Let $k \in \{0, 1, ..., n-1\}$ be the least such that $r^k \in L_R$. Then the vertex $r^k \in L_R$ is the Rotation Triumph Vertex, which we will denote t_R. Let $l \in \{1, ..., n\}$ be the most such that $r^l s \in L_F$. Then the vertex $r^l s \in L_F$ is the Reflection Triumph Vertex, which we will denote t_F.*

For the remainder of the chapter we will use t_R and t_F to denote the Rotation Triumph Vertex and the Reflection Triumph Vertex, respectively. We define $t_R = r^k$ as *even* if k is even and t_F as *odd* if k is odd. We define $t_F = r^l s = r^l s^1$ to be *even* if $l+1$ is even and t_F to be *odd* if $l+1$ is odd.

Lemma 3.3. *Let the Player move to r^i where $w_{r^{i+1}} > w_{r^{i+1}s}$. Then the Player wins if the Opponent reduces w_{r^i} such that $w_{r^i} < w_{r^i s}$. Similarly, let the Player move to $r^i s$ where $w_{r^{i-1}s} > w_{r^{i-1}}$. Then the Player wins if the Opponent reduces $w_{r^i s}$ such that $w_{r^i s} < w_{r^i}$.*

Proof. Suppose the Opponent reduces w_{r^i} such that $w_{r^i} < w_{r^i s}$. The Player forces the Opponent to move to r^{i+1} by Lemma 3.2.

Consider this a new game in which the Player starts on r^{i+1} and the current weights of the vertices are now the initial weights for this new game. Let w_m be the minimum of $\{w_{r^i,0}, w_{r^{i+1},0}, w_{r^{i+1}s,0}, w_{r^i s,0}\}$. Since $w_{r^i,0} < w_{r^i s,0}$ and $w_{r^{i+1},0} > w_{r^{i+1}s,0}$, then we know $w_m = w_{r^i}$ or $w_m = w_{r^{i+1}s}$. We will induct on w_m. For the initial case, suppose $w_m = 1$.

The Player reduces $w_{r^{i+1},0}$ by 1 and moves to $r^{i+1}s$. Now $w_{r^{i+1}s,0} \leq w_{r^{i+1}}$. If $w_m = w_{r^{i+1}s}$, then the Opponent must reduce w_m to 0 and move to $r^i s$ to avoid defeat by Lemma 3.1. If $w_m \neq w_{r^{i+1}s}$, the Opponent reduces $w_{r^{i+1}s,0}$ such that $w_{r^{i+1}s} < w_{r^{i+1}}$ and is forced to move to $r^i s$ by Lemma 3.2. Then the Player reduces $w_{r^i s,0}$ by 1 and moves to r^i. If $w_m = w_{r^i}$, the Opponent reduces w_m to 0, and the Opponent moves to r^{i+1} to avoid defeat by Lemma 3.1. Otherwise, the Opponent reduces $w_{r^i,0}$ such that $w_{r^i} < w_{r^i s}$ and is forced to move to r^{i+1} by Lemma 3.2. Now we are back to the initial vertex where either $w_{r^{i+1s}} = 0$ or $w_{r^i} = 0$. The Player reduces $w_{r^{i+1}}$ by 1 and moves to $r^{i+1}s$. If $w_{r^{i+1}s} = 0$, the Opponent is unable to reduce $w_{r^{i+1}s}$, and the Player wins. Otherwise, the Opponent reduces $w_{r^{i+1}s}$ and is forced to move to $r^i s$ by Lemma 3.2. The Player reduces $w_{r^i s}$ by 1 and moves to r^i. Since $w_{r^i} = 0$, the Opponent is unable to reduce w_{r^i}, and the Player wins. Notice that since we started with $w_{r^i,0} < w_{r^i s,0}$ and $w_{r^{i+1}s,0} < w_{r^{i+1},0}$, and since Player 1 always reduces by 1, then it is always true that $w_{r^i} \leq w_{r^i s}$ and $w_{r^{i+1}s} \leq w_{r^{i+1}}$.

For the induction step, assume the Player wins for $1 \leq w_m \leq k$. Consider $w_m = k+1$. The Player reduces $w_{r^{i+1},0}$ by 1 and moves to $r^{i+1}s$. If $w_m = w_{r^{i+1}s}$, then the Opponent reduces $w_{m,0}$ such that $w_m \leq w_{m,0} - 1 = (k+1) - 1 = k$, and the Opponent moves to $r^i s$ to avoid defeat by Lemma 3.1. Otherwise, the Opponent reduces $w_{r^{i+1}s,0}$ such that $w_{r^{i+1}s} < w_{r^{i+1}}$ and is forced to move to $r^i s$ by Lemma 3.2. Then the Player reduces $w_{r^i s,0}$ by 1 and moves to r^i. If $w_m = w_{r^i}$, the Opponent reduces w_m such that $w_m \leq w_{m,0} - 1 = (k+1) - 1 = k$, and the Opponent moves to r^{i+1} to avoid defeat by Lemma 3.1. Otherwise, the Opponent reduces $w_{r^i,0}$ such that $w_{r^i} < w_{r^i s}$ and is forced to move to r^{i+1} by Lemma 3.2. Now we are back at the starting vertex where either $w_{r^{i+1}s} \leq k$ or $w_{r^i} \leq k$. Then the Player wins by the induction hypothesis.

Suppose the Opponent reduces $w_{r^i s}$ such that $w_{r^i s} < w_{r^i}$. Then the Player wins by similar induction. □

Lemma 3.3 lets us assume a player will avoid defeat by keeping $w_{r^i} \geq w_{r^i s}$ whenever $w_{r^{i+1}} > w_{r^{i+1}s}$ and keeping $w_{r^i s} \geq w_{r^i}$ whenever $w_{r^{i-1}s} > w_{r^{i-1}s}$. This reduces the number of ways a player can reduce the weight of a vertex before moving, which is more efficient when determining winning strategies. The following lemma relates Triumph vertices to Lemma 3.3. While we just saw what a player should avoid doing, the next lemma is about what a player should do.

Lemma 3.4. *A Player is guaranteed victory if he or she moves to t_R. Similarly, the Player is guaranteed victory if he or she moves to t_F.*

Proof. Let the Player move to t_R. The Opponent reduces $w_{t_R,0}$ such that $w_{t_R} < w_{r^i s,0}$ and is forced to move to r^{i+1} by Lemma 3.2. Then the Opponent loses by Lemma 3.3.

Then the Player wins if he or she moves to t_F by similar reasoning.

□

Since moving to a Triumph Vertex guarantees victory, we see how they are named appropriately. We refer to this strategy as the Triumph Cycle Strategy where a *Triumph Cycle* is defined as the sequence of vertices used in determining a Triumph Vertex. The sequence

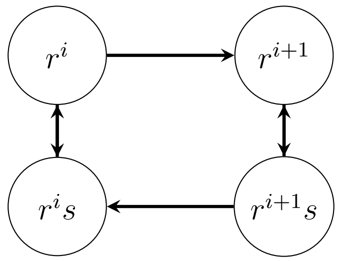
$$r^i, r^{i+1}, r^{i+1}s, r^i s$$

is a Triumph Cycle as long as $r^i \in L_R$. Another Triumph Cycle is

$$r^i s, r^{i-1} s, r^{i-1}, r^i$$

as long as $r^i s \in L_F$.

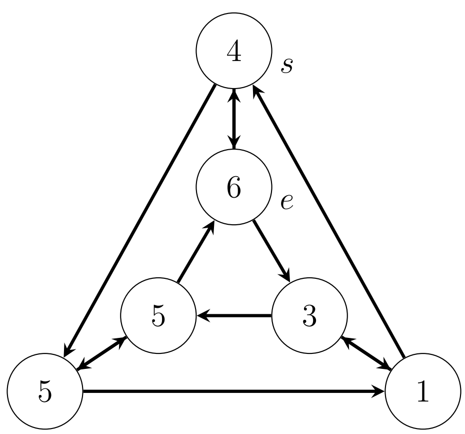
Example 6. *A Triumph Cycle is illustrated below.*



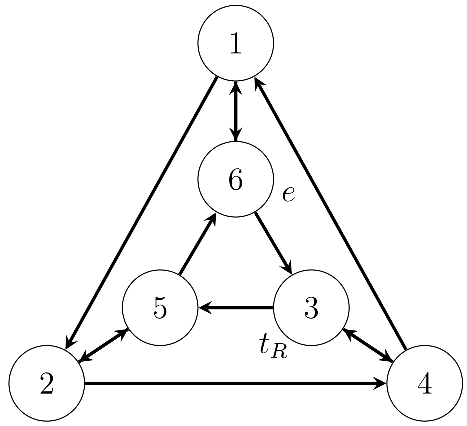
$$w_{r^i} \le w_{r^i s}$$
$$w_{r^{i+1}} > w_{r^{i+1} s}$$

Triumph Cycles affect the game in two central ways. The Player is able to create a Triumph Cycle through general game play; however, this results in victory for the Opponent, as proved by Lemma 3.3. The Player is also able to move into a Triumph Cycle via the Triumph Vertex, which would result in victory as proved by Lemma 3.4. Both of these situations are illustrated in the following examples.

Example 7. *If Player 1 reduces the identity so $w_e < w_s$, the Player 2 wins by the Triumph Cycle consisting of vertices with initial weights {6,3,1,4}.*



Example 8. *Player 1 can move to the Rotation Cycle Triumph Vertex and win along the Triumph Cycle consisting of vertices with initial weights {3,5,2,4 }.*



Thus Triumph Cycles can be both beneficial and detrimental to a player's game play. These two strategies, Forced Play and Triumph Cycle, create the foundation from which we deduce winning strategies for Nim on dihedral groups.

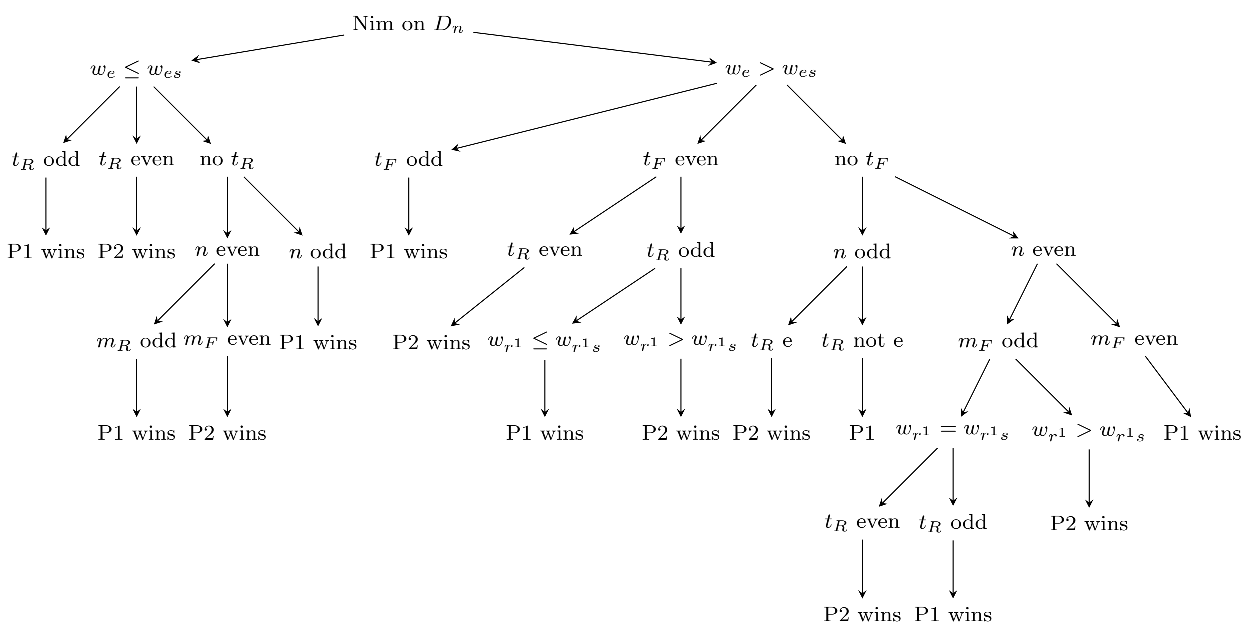
3.3 Strategic Explanation

The strategies for Nim on D_n, like Nim on cyclic groups, depend exclusively on the initial conditions of the gameboard.

We first look at the Triumph Vertices, t_R and t_F, because if the Player moves to one of these vertices, the Player wins the game. The next step is to determine whether or not the Player can actually reach the Triumph Vertices based on the Opponent's moves in the game. There are two variables to determine if this happens. The first is the placement of the Triumph Vertices. If the Triumph Vertex is even, only Player 2 has a chance of moving to it. If the Triumph Vertex is odd, only Player 1 has a chance of moving to it. The next variable is the relationship between the initial weights of the vertices in the Rotation Cycle and the Reflection Cycle. This will determine whether the Opponent can successfully switch cycles before the Player reaches a Triumph Vertex. Note if $t_R = r^k$, then $w_{r^i} \leq w_{r^is}$ for all $i \leq k$. Similarly, if $t_F = r^ks$, then $w_{r^is} \leq w_{r^i}$ for all $i \geq k$. These conditions follow from the way we defined t_R and t_F to be the vertices in the Triumph Sets which are the closest to the identity vertex. Once the Player reaches a Triumph Vertex, the Player wins the game along the Triumph Cycle.

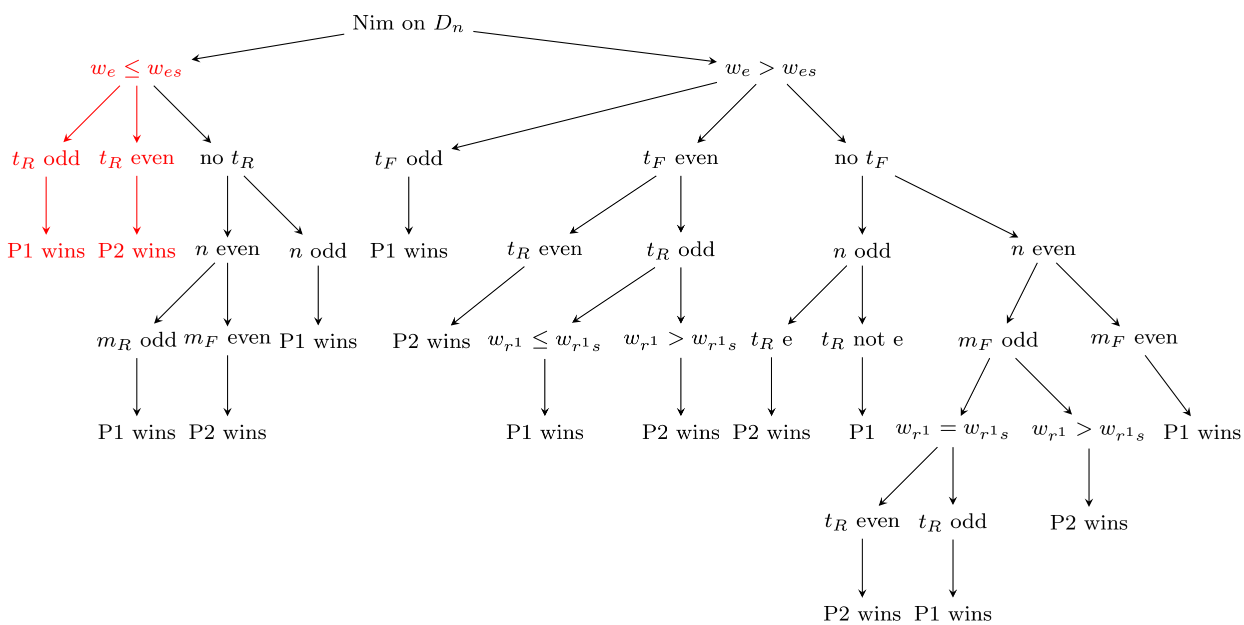
If there happens to be no Triumph Vertex, we look at the vertices with the minimum initial weight. Again, we look at whether the minimum vertex is even or odd, and then we look at the relationship between the initial weights of vertices along the Rotation Cycle and the vertices along the Reflection Cycle to determine whether the Player can move to these minimum weight vertices. Then the Player can use the Forced Player Strategy to force the Opponent to reduce the weights of the minimum vertices to win the game. This is the basic way we determine which player has a winning strategy.

Given any initial weight distribution for any game of Nim on D_n, there exists a winning strategy for either Player 1 or Player 2. As long as the Player with the winning strategy moves according to defined strategy, the Player is guaranteed victory. The following diagram outlines all possible initial game conditions identifying which player has the winning strategy. In the next section, we will prove each part of the diagram.



3.4 Winning Strategy

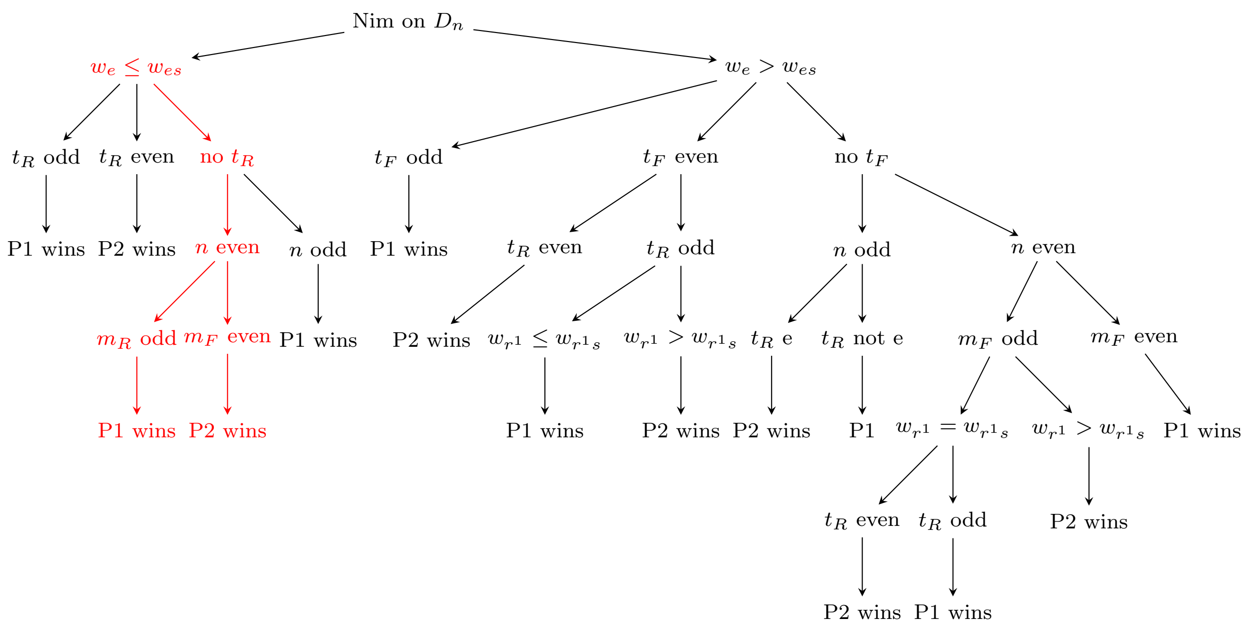
Now we look at proofs of all possible Nim games on dihedral groups. Each theorem will prove a section of the chart, which will be highlighted in red.



Theorem 3.1. *Let G be the Cayley graph for D_n. Let $w_e \leq w_{es}$ and $t_R \in G$. If t_R is odd, Player 1 has the winning strategy. If t_R is even, Player 2 has the winning strategy.*

Proof. Let $w_e \leq w_{es}$ and $t_R = r^k \in G$. Suppose t_R is odd. Player 1's strategy is to reduce the even vertices by 1 and move to the next odd vertex along the Rotation Cycle. We know $w_{r^j} \leq w_{r^j s}$ for $0 \leq j \leq k$ since $t_R = r^k$. Then Player 1 can force Player 2 to continue on the Rotation Cycle for all r^j by Lemma 3.2. Since t_R is odd, Player 1 will eventually move to t_R. Then Player 1 continues along the Triumph Cycle and wins by Lemma 3.4.

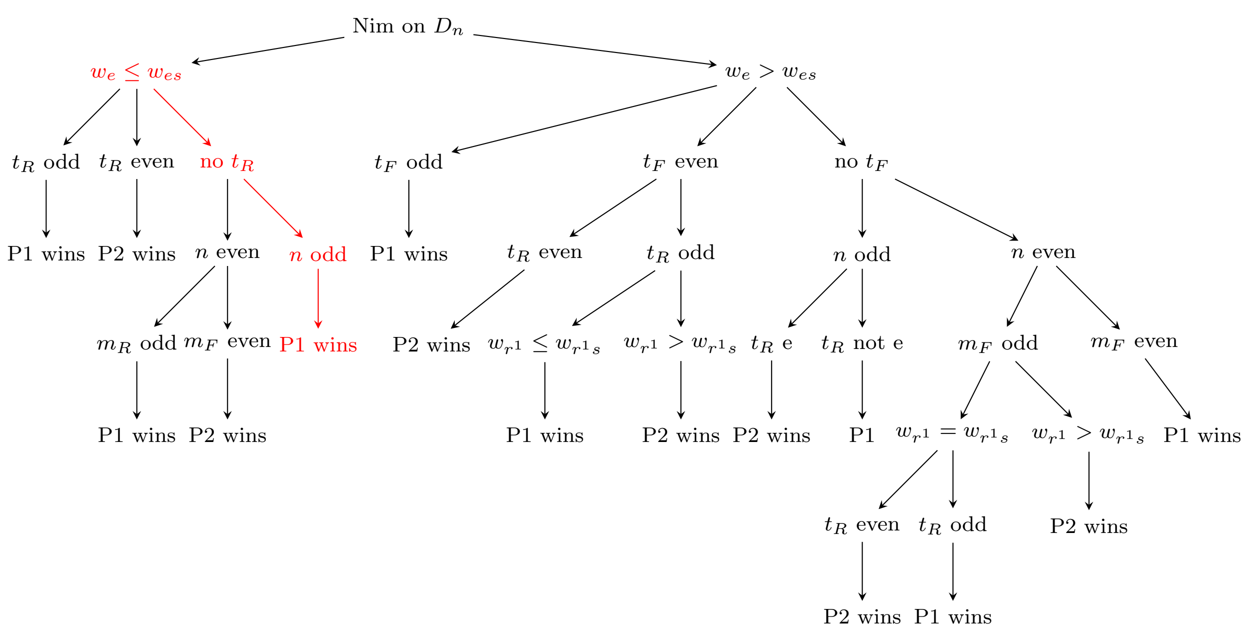
Suppose t_R is even. Player 2's strategy is to reduce the odd vertices by 1 and move to the next even vertex. Since $w_{r^j} \leq w_{r^j s}$, Player 2 can force Player 1 to continue along the Rotation Cycle for all r^j by Lemma 3.2. Since t_R is even, Player 2 will eventually move to t_R. Then Player 2 continues along the Triumph Cycle and wins by Lemma 3.4. $\square$



In order for t_R to exist, there must be at least one vertex such that $w_{r^i} > w_{r^i s}$. Since Theorem 3.2 presumes $w_{r^i} \leq w_{r^i s}$ for all $r^i, r^i s \in G$, it is implied that there is no t_R in this particular game situation. Also note that in Theorem 3.2, r^k corresponds to m_R in the diagram.

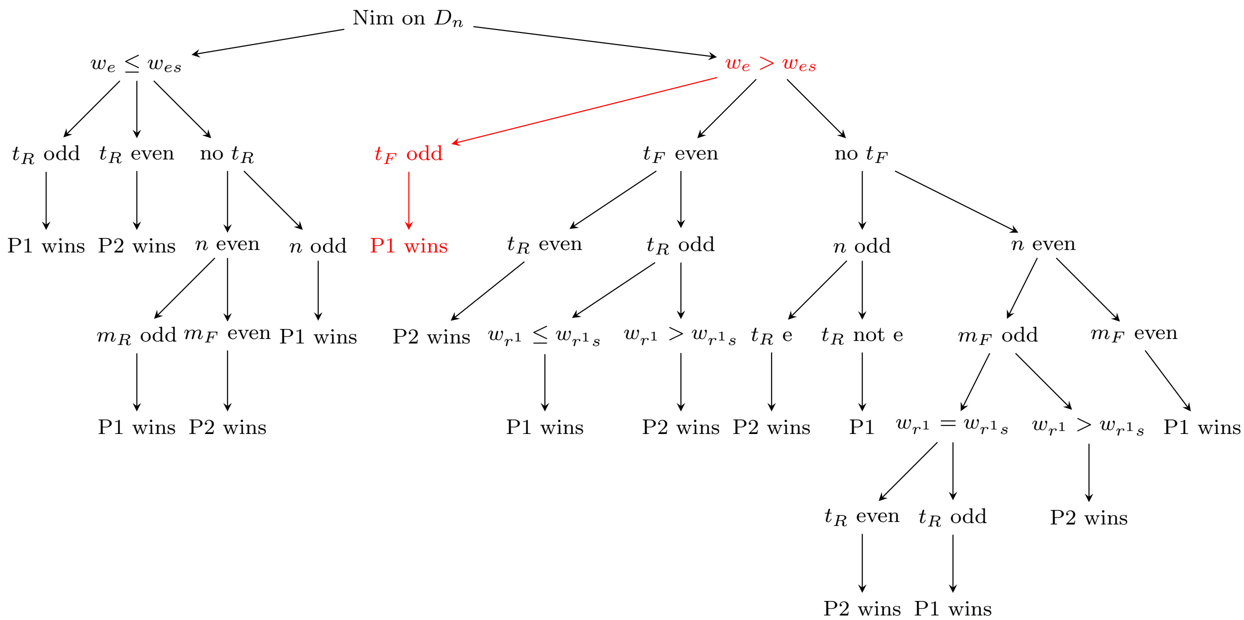
Theorem 3.2. *Let G be the Cayley graph for D_n where n is even. Let $w_{r^i} \leq w_{r^i s}$ for all $r^i, r^i s \in G$. Let $\min\{w_e, w_{r^1}, ..., w_{r^{n-1}}\} = w$. Let $k \in \{0, 1, ..., n-1\}$ be the least such that $w_{r^k} = w$. Then if k is even, Player 2 has the winning strategy. If k is odd, Player 1 has the winning strategy.*

Proof. Since $w_{r^i} \leq w_{r^i s}$, each player can confine the other to the Rotation Cycle by Lemma 3.2. Then the game reduces to $\mathbb{Z}_n$ and Player 2 wins if k is even and Player 1 wins if k is odd by Theorem 2.2. $\square$



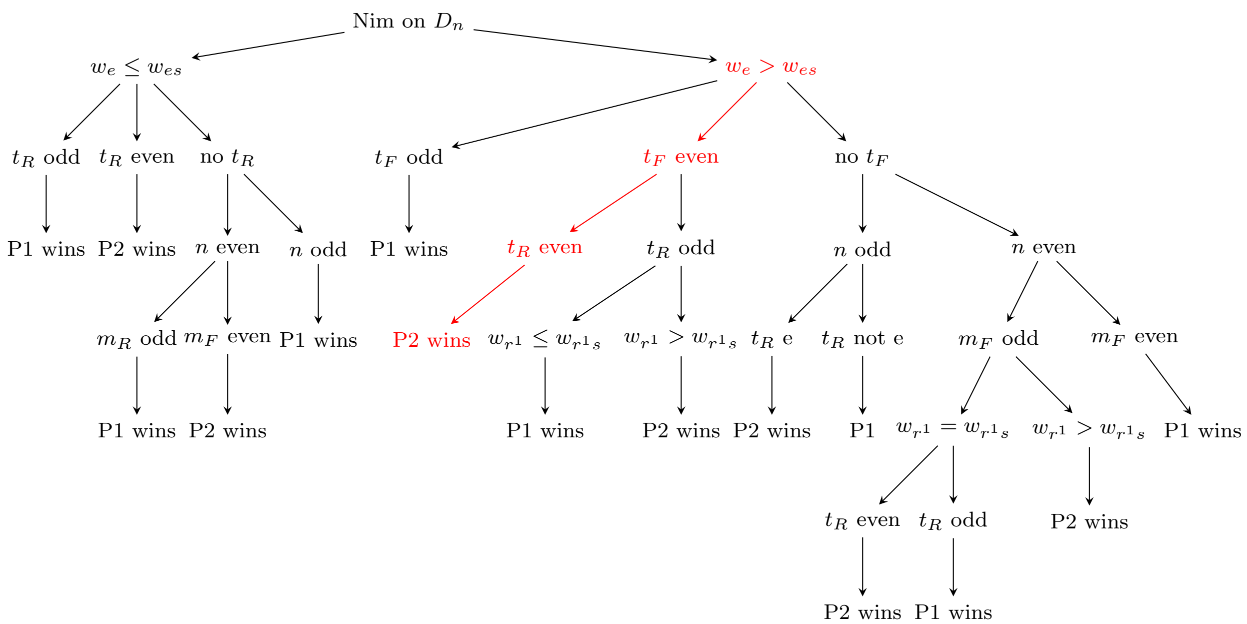
Theorem 3.3. *Let G be the Cayley graph for D_n when n is odd and $w_{r^i} \leq w_{r^i s}$ for all $r^i, r^i s \in G$ where i is odd. Then Player 1 has the winning strategy.*

Proof. Player 1 first reduces w_e to 0 and continues play along the Rotation Cycle. Since $w_{r^i} \leq w_{r^i s}$ for all odd i, Player 1 forces Player 2 along the Rotation Cycle by the Forced Play Strategy of Lemma 3.2. Then the game reduces to $\mathbb{Z}_n$, and Player 1 has the winning strategy by Theorem 2.1. $\square$



Theorem 3.4. *Let G be the Cayley graph for D_n and $w_e > w_{es}$. If $t_F \in G$ is odd, Player 1 has the winning strategy.*

Proof. Let $t_F = r^k s \in G$ be odd. Player 1 starts by reducing w_e by 1 and moving to w_{es}. Since $t_F = r^k s$, we know k is the greatest integer such that $w_{r^{k-1}s} > w_{r^{k-1}}$. Thus $w_{r^j s} \leq w_{r^j}$ for $k \leq j \leq n$. Then Player 1 can force Player 2 to move along the Reflection Cycle by the Forced Play Strategy of Lemma 3.2. Since t_F is odd, Player 1 will eventually move to t_F. Then Player 1 continues along the Triumph cycle and wins by Lemma 3.4. $\square$



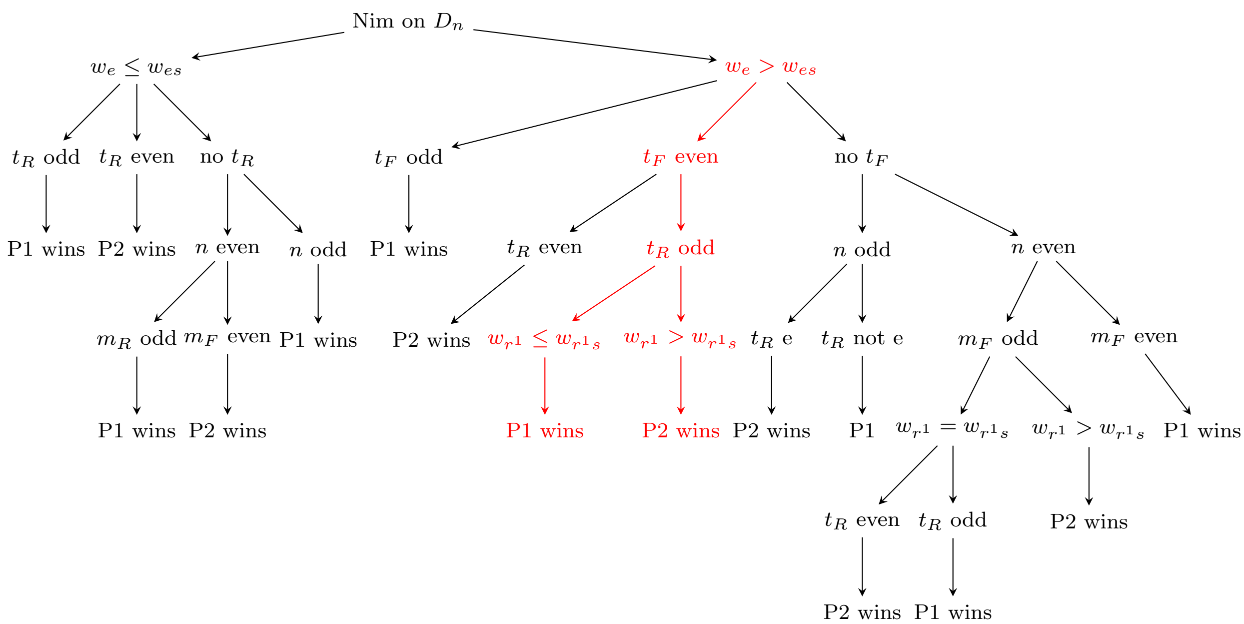
Theorem 3.5. *Let G be the Cayley graph for D_n and $w_e > w_{es}$. Suppose there exists $t_F \in G$, which is even. Let $t_R \in G$. If t_R is even, then Player 2 wins.*

Proof. Let $t_F = r^k s \in G$ be even. Suppose t_R is even. Player 1 starts by reducing w_e by some amount and moving to either es or r^1. If Player 1 moves to es, Player 2's strategy is to force Player 1 along the Reflection Cycle by the Forced Play Strategy of Lemma 3.2. Since t_F is even, Player 2 wins by moving to t_F and continuing along the Triumph Cycle by Lemma 3.4.

Now suppose Player 1 moves to r^1. If $w_{r^1} > w_{r^1s}$, then Player 1 must set $w_e \geq w_{es}$ to avoid losing by Lemma 3.3. Player 2 reduces w_{r^1} by 1 and moves to r^1s. Then Player 2 can force Player 1 along the Reflection Cycle by Lemma 3.2. Since t_F is even, Player 2 will eventually move to t_F. Then Player 2 continues along the Triumph Cycle and wins by Lemma 3.2.

If $w_{r^1} \leq w_{r^1s}$, then Player 2 reduces w_{r^1} by 1 and moves to r^2. Player 2 forces play along the Rotation Cycle by Lemma 3.2. Since t_R is even, Player 2 moves to t_R and wins by Lemma 3.4.

□



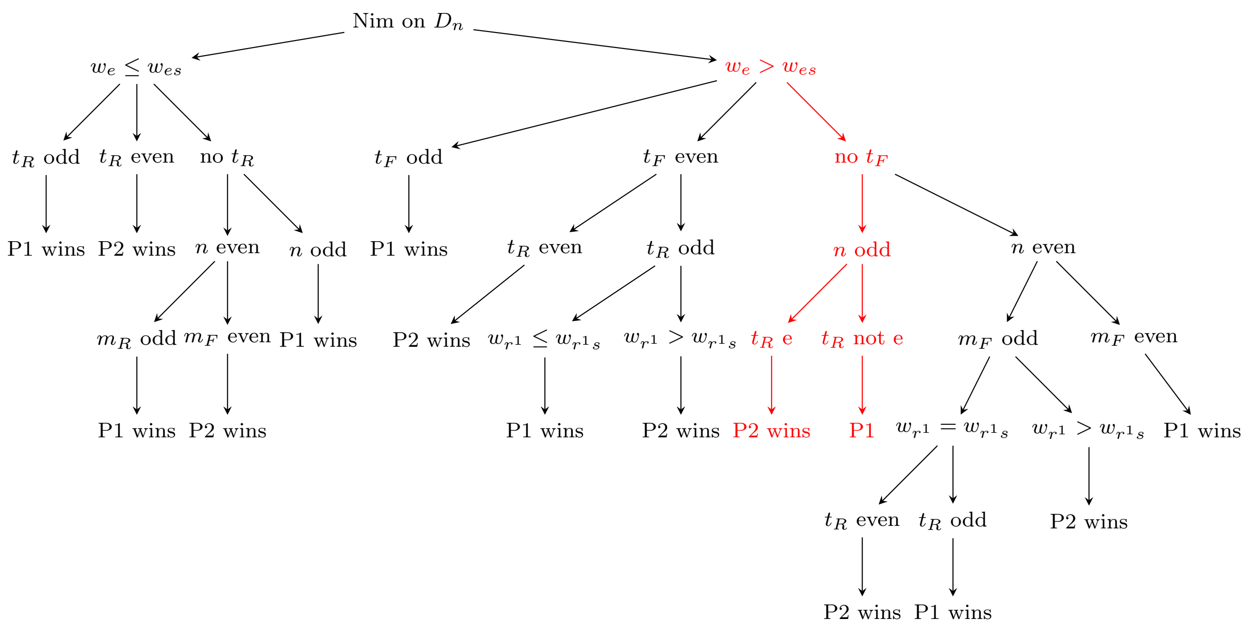
Theorem 3.6. *Let G be the Cayley graph for D_n and $w_e > w_{es}$. Suppose there exists $t_F \in G$, which is even. Let $t_R \in G$. If t_R is odd then Player 1 wins if $w_{r^1} \leq w_{r^1 s}$ and Player 2 wins if $w_{r^1} > w_{r^1 s}$.*

Proof. Let $t_F = r^k s \in G$ be even. Suppose t_R is odd and $w_{r^1} \leq w_{r^1 s}$. Player 1 starts by reducing w_e by 1 and moving to w_{r^1}. Since $w_{r^1} \leq w_{r^1 s}$, we know $w_{r^j} \leq w_{r^j s}$ for $1 \leq j \leq k$. Then Player 1 can force Player 2 to continue on the Rotation Cycle for all r^j by the Forced Play Strategy of Lemma 3.2. Since t_R is odd, Player 1 will eventually move to t_R. Then Player 1 continues along the Trumph Cycle and wins by Lemma 3.4.

Now suppose $w_{r^1} > w_{r^1 s}$. Player 1 starts by reducing w_e by some amount and moving to either es or r^1. If Player 1 moves to es, then Player 2 will move along the Reflection Cycle. Since $w_{r^j s} \leq w_{r^j}$ for $k \leq j \leq n$, Player 2 can force Player 1 along the Reflection Cycle by Lemma 3.2. Since t_F is even, Player 2 will eventually move to t_F. Then Player 2 continues along the Triumph Cycle and wins by Lemma 3.2.

Suppose Player 1 moves to r^1. Player 1 must leave $w_e \geq w_{es}$ by Lemma 3.3. Then Player 2 reduces w_{r^1} by 1 and moves to $r^1 s$ on the Reflection Cycle. Now $w_{r^1 s} \leq w_{r^1}$, $w_{es} \leq w_e$, and $w_{r^j s} \leq w_{r^j}$ for $k \leq j \leq n$, and Player 2 wins by the above argument.

□



In order for t_F to exist, there must be at least one vertex such that $w_{r^i s} > w_{r^i}$. Since Theorem 3.2 presumes $w_{r^i s} \leq w_{r^i}$ for all $r^i, r^i s \in G$, it is implied that there is no t_F in this particular game situation. Also notice that in Theorem 3.7 we refer to the case where t_R exists. We will discuss the no t_R case in Theorem 3.8.

Theorem 3.7. *Let G be the Cayley graph for D_n where n is odd. Suppose $w_e > w_{es}$ and $w_{r^i,0} \geq w_{r^i s,0}$ for all $1 \leq i \leq n-1$. Let $t_R \in G$. Then Player 2 has the winning strategy if t_R is even and Player 1 has the winning strategy if t_R is odd.*

Proof. Suppose $t_R = r^k \in G$ is even. Player 1 starts by moving to either es or r^1. If Player 1 moves to es we have a new game in which $w_{r^i s} \leq w_{r^i}$ for $1 \leq i \leq n-1$. We see this game is equivalent to that of Theorem 3.3, the only difference being Player 2 wins along the Reflection Cycle instead of the Rotation Cycle. Thus, Player 1 will move to r^1 to avoid defeat. Also note that Player 1 must reduce $w_e \geq w_{es,0}$ if $w_{r^1} > w_{r^1 s}$ to avoid losing by Lemma 3.3. (If $w_{r^1} = w_{r^1 s}$, then how Player 1 reduces w_e is nonapplicable to the outcome of the game.)

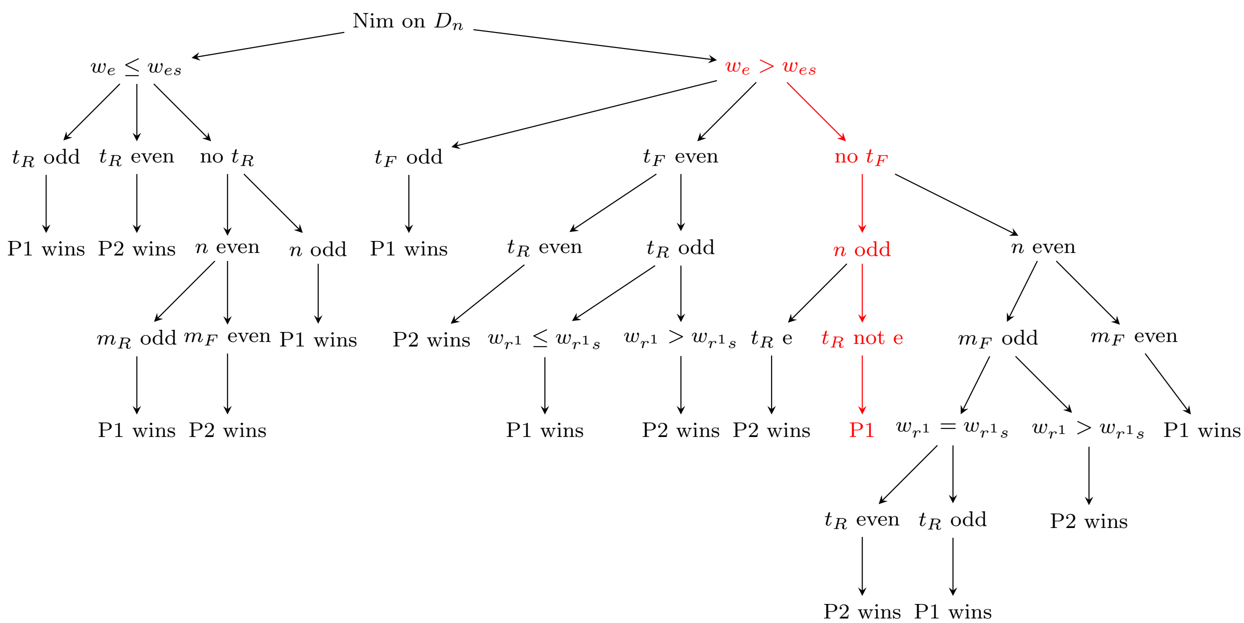
Player 2's strategy is to reduce the weight of all odd vertices by 1 and continue along the Rotation Cycle. At each even vertex r^j, there are 2 possible scenarios for Player 1.

Case 1: $w_{r^j,0} = w_{r^j s,0}$. Then Player 2 can force Player 1 to continue along the Rotation Cycle by Lemma 3.2.

Case 2: $w_{r^j,0} > w_{r^j s,0}$. Since $t_R = r^k$, then $w_{r^i} \geq w_{r^i s}$ for all $0 \leq i \leq j$ since the players reduce to avoid defeat by Lemma 3.3. Then if Player 1 moves from r^j to $r^j s$, we have a game that is equivalent to that of Theorem 3.3 as seen above.

Then play continues along the Rotation Cycle. Since t_R is even, Player 2 will move to t_R and wins along the Triumph Cycle by Lemma 3.4.

Suppose t_R is odd. Player 1's strategy is to reduce the weight of the even vertices by 1 and continue along the Rotation Cycle to eventually reach t_R. Player 1 will reach t_R by similar reasoning from above. $\square$



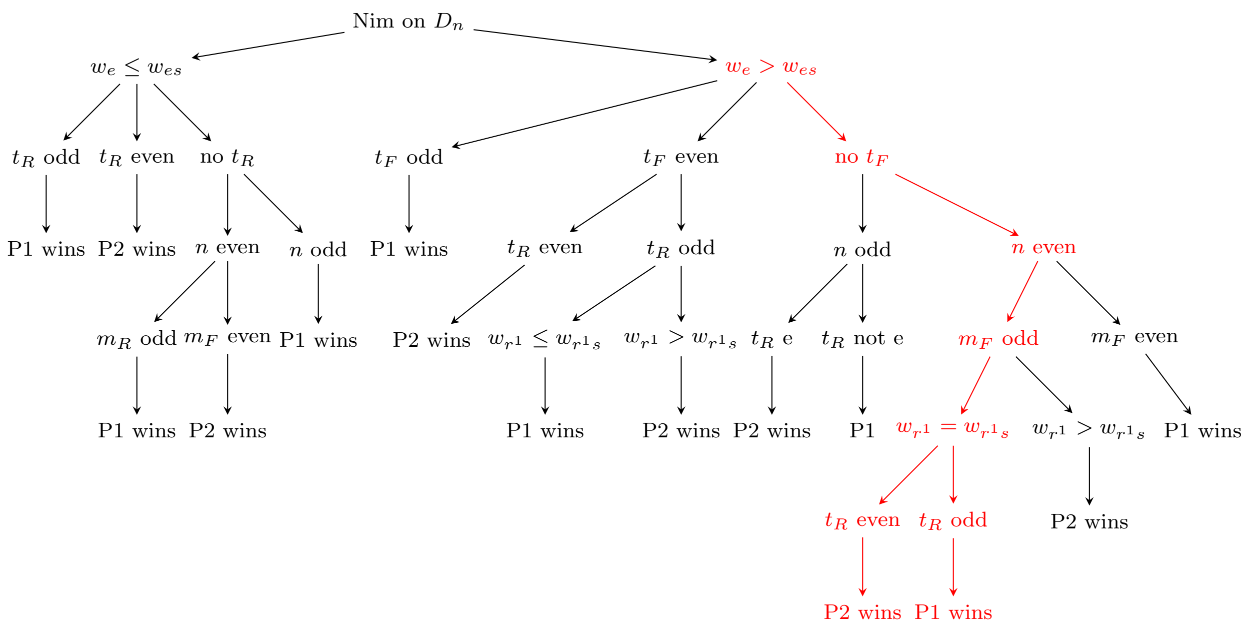
In Theorem 3.8, there is no t_R and no t_F since all the weights on the Rotation Cycle are strictly greater than all the corresponding weights on the Reflection Cycle.

Theorem 3.8. *Let G be the Cayley graph for D_n where n is odd. Suppose $w_{r^i,0} > w_{r^i s,0}$ for all $r^i, r^i s \in G$. Then Player 1 has the winning strategy.*

Proof. Player 1 reduces $w_{e,0}$ such that $w_e = w_{es,0}$ and moves to r^1. Player 1's strategy is to reduce the weight of the even vertices r^j such that $w_{r^j} = w_{r^j s}$ and move along the Rotation Cycle to r^{j+1}. The remainder of the strategy depends on Player 2's choices at r^i where i is odd. We may assume Player 2 reduces $w_{r^i,0}$ such that $w_{r^i} \geq w_{r^i s,0}$ to avoid creating a Triumph Cycle for Player 1 to win as seen in Lemma 3.3. Then Player 2 can either move to $r^i s$ or r^{i+1}.

Suppose Player 2 switches cycles and moves to $r^i s$. Player 1 reduces $w_{r^i s,0}$ to 0. Notice we have that $w_{r^k s} \leq w_{r^k}$ for $0 \leq k \leq n-1$. Thus Player 1 wins by Theorem 3.3. The only difference is that in this case Player 1 wins along the Reflection Cycle, which is equivalent to the Rotation Cycle of Theorem 3.3. Then we may assume Player 2 always moves to r^{i+1} to avoid defeat.

Play continues until Player 2 moves to the even vertex r^{n-1}. Now Player 1 reduces $w_{r^{n-1},0}$ to 0. We have $w_{r^n} = w_{r^n s}, w_{r^2} = w_{r^2 s}, \ldots, w_{r^{n-2}} = w_{r^{n-2} s}$ because of Player 1's strategy up to this point. If we consider the new game in which r^{n-1} is the initial vertex, we see this game fulfills the hypothesis of Theorem 3.3. Then Player 1 has the winning strategy. $\square$



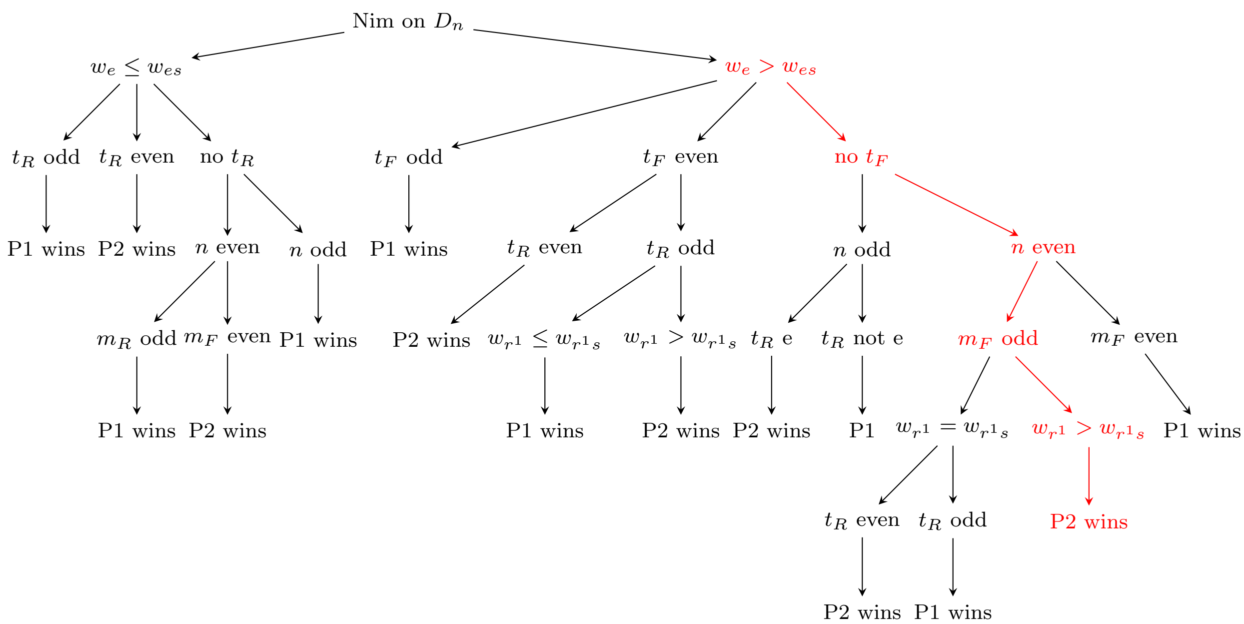
Notice if $w_{r^1} < w_{r^is}$ and $w_{r^i} \geq w_{r^is}$ for $2 \leq i \leq n$, then we see $t_F = r^2s$ since $w_{r^2s} \leq w_{r^2}$ and $w_{r^1s} > w_{r^1}$. Since the following two theorems assume there is no t_F, we see the case where $w_{r^1} < w_{r^1s}$ is non applicable, and therefore, we do not need to include it.

Theorem 3.9. *Let G be the Cayley graph for D_n where n is even. Let $w_e > w_{es}$, $w_{r^1} = w_{r^1}$, and $w_{r^i} \geq w_{r^is}$ for $2 \leq i \leq n-1$. Let $t_R \in G$ and $\min\{w_{r^0s}, w_{r^1s}, ..., w_{r^{n-1}s}\} = w$. Let $m_F \in \{1, 2, ..., n\}$ be the most such that $w_{r^{m_F}s} = w$. Suppose m_F is odd. Then if t_R is odd, Player 1 has the winning strategy, and if t_R is even, Player 2 has the winning strategy.*

Proof. Suppose $t_R = r^k \in G$ is odd. Player 1's strategy is to reduce the weight of the even vertices r^i by 1 and continue along the Rotation Cycle. So Player 1 starts by reducing w_e by 1 and moving to w_{r^1}. Since $w_{r^1} = w_{r^1s}$, we know $w_{r^j,0} = w_{r^js,0}$ for $1 \leq j \leq k$. Then Player 1 can force Player 2 to continue on the Rotation Cycle for all r^j by Lemma 3.2. Since t_R is odd, Player 1 will eventually move to t_R and win by Lemma 3.4.

Suppose t_R is even. Player 1 either moves to r^1 or es. If Player 1 moves to r^1, Player 2 moves to r^2 and forces Player 1 to continue along the Rotation Cycle by Lemma 3.2 since $w_{r^j,0} = w_{r^js,0}$ for $1 \leq j \leq k$. Since t_R is even, Player 2 eventually moves to t_R and wins along the Triumph cycle by Lemma 3.4.

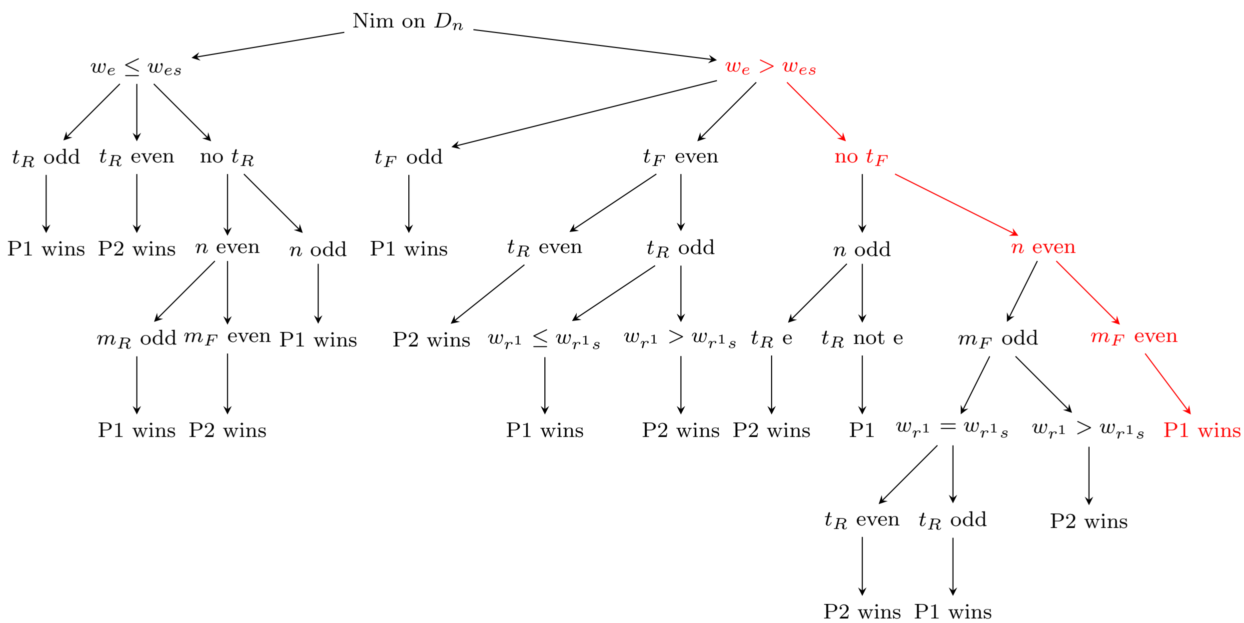
Suppose Player 1 moves to es. If Player 1 reduced e such that $w_e < w_{es}$, Player 2 can move back to e and can force Player 1 to continue along the Rotation Cycle by Lemma 3.2. Assume Player 1 sets $w_e \geq w_{es}$ and moves to es. Since $w_{r^is} \leq w_{r^i}$ for all $r^is, r^i \in G$, the game is reduced to a game on $\mathbb{Z}_n$. Player 2's strategy is to force Player 1 to continue along the Reflection Cycle by Lemma 3.2. Since m_F is odd, Player 2 has the winning strategy by Theorem 2.2. $\square$



Theorem 3.10. *Let G be the Cayley graph for D_n where n is even. Let $w_e > w_{es}$, $w_{r^1} > w_{r^1s}$, and $w_{r^i} \ge w_{r^is}$ for $2 \le i \le n-1$. Let w be the minimum of $\{w_{es}, w_{r^1s}, ..., w_{r^{n-1}s}\}$. Let $m_F \in \{1, 2, ..., n\}$ be the most such that $w_{r^{m_F}s} = w$. If m_F is odd, Player 2 has the winning strategy.*

Proof. Suppose m_F is odd. Player 1 reduces w_e and can either move to es or r^1. Suppose Player 1 moves to es. Then the game is reduced to $\mathbb{Z}_n$ since Player 2 can force Player 1 to continue along the Reflection Cycle. Player 2's strategy is to reduce the weight of each vertex by 1 and continue along the Reflection Cycle. Since m_F is odd, Player 2 has the winning strategy by Theorem 2.2.

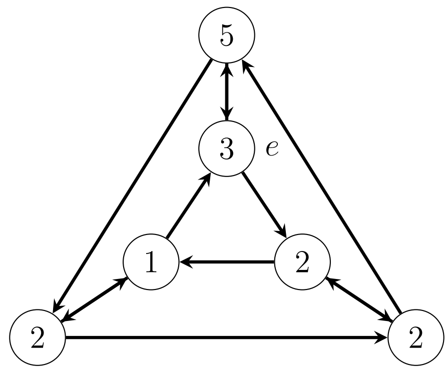
Suppose Player 1 moves to r^1. We assume Player 1 sets $w_e \ge w_{es}$ to avoid defeat by Lemma 3.3. Then Player 2 reduces w_{r^1} by 1 and switches cycles to r^1s. Since $w_{r^is} \le w_{r^i}$ for all $r^is, r^i \in G$, then again Player 2 has the winning strategy by Theorem 2.2. $\square$



Theorem 3.11. *Let G be the Cayley graph for D_n where n is even. Let $w_e > w_{es}$ and $w_{r^i} \geq w_{r^is}$ for $1 \leq i \leq n-1$. Let w be the minimum of $\{w_{es}, w_{r^1s}, ..., w_{r^{n-1}s}\}$. Let $m_F \in \{1, 2, ..., n\}$ be the most such that $w_{r^{m_F}s} = w$. If m_F is even, Player 1 has the winning strategy.*

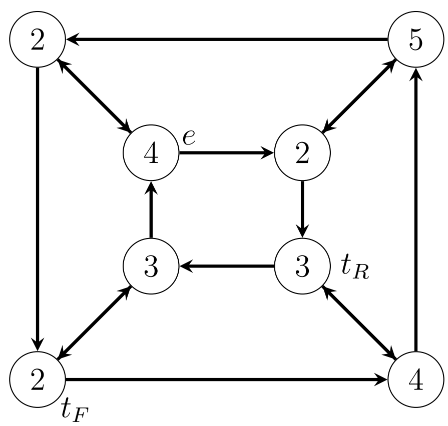
Proof. Suppose m_F is even. Player 1 reduces w_e by 1 and moves to es. Then we have $w_{r^is} \leq w_{r^i}$ for all $r^i, r^is \in G$. The game is then reduced to $\mathbb{Z}_n$ since Player 1 can force Player 2 to continue along the Reflection Cycle by Lemma 3.2. Then since m_F is even, Player 1 has the winning strategy by Theorem 2.2. $\square$

Example 9. *Sample Nim Game*



Player 1 will trap Player 2 on the identity vertex e in the above example according to Theorem 3.3.

Example 10. *Sample Nim Game*



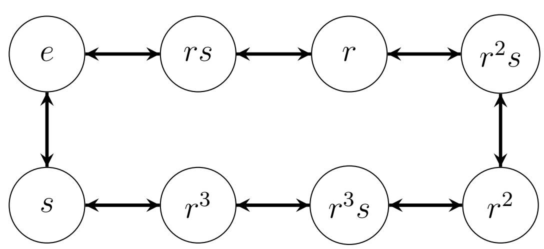
Player 2 wins on the game above according to Theorem 3.4 since t_F is even and t_R is even.

Chapter 4

Finite Groups of Two Involutions

We now consider Nim on Cayley graphs of dihedral groups with generators other than r and s. Consider D_4 generated by rs and s, pictured below. The Cayley graph is a cycle of length 8 with double sided arrows instead of the single arrows we saw in the cyclic groups $\mathbb{Z}_n$. In fact, this is true for all other generating sets for D_n of order 2. These include $\{rs, s\}, \{r^2s, rs\}, \{r^3s, r^2s\}, \{r^3s, s\}$. Notice these generators are *involutions*, which means for each generator g, $xgg = x$ for all $x \in D_n$. The Cayley graph of D_n generated by two involutions is isomorphic to the Cayley graph of $\mathbb{Z}_n$, the only difference being double sided arrows connect the vertices rather than the single sided arrows we dealt with in Chapter 2.

Example 11. *The Cayley graph of D_4 generated by rs and s.*



All groups generated by two involutions will have Cayley graphs similar to that above. We generalize our results from dihedral groups to obtain all winning strategies for Nim on all groups of two involutions. The basic strategy is for Player 1 to choose to move either clockwise or counterclockwise depending on where the advantage lies. The only way Player 2 can win is if Player 2 has the advantage from both sides. The exact strategies are outlined below.

Consider the Cayley graph of a group G of order n with generating set $S = \{g : xgg = x, \forall x \in G\}$ such that $|S| = 2$. Label the n vertices of G as $\{0, 1, 2, ..., n-1\}$ where i is connected to $i+1$ for $i = 0$ to $i = n-1$, and $n-1$ is connected to 0. An odd vertex $x \in G$ is a vertex labeled with an odd number and an even vertex $x \in G$ is a vertex labeled with an even number. As defined previously, $w_{x,0}$ will refer to the initial weight of the vertex x and w_x will refer to weight of vertex x at any other point in the game.

Theorem 4.1. *Let G be the Cayley graph of any group generated by two involutions where $|G| = n$. Let x be the least such that $w_{x,0} \leq w_{x+1,0}$ for $x \in G$. If x is odd, then Player 1 has the winning strategy.*

Proof. Suppose x is odd. Let $w_{x,0} = m$. Then $w_{i,0} > w_{i+1,0}$ for $0 \leq i < x$. We will induct on m. For the initial case, let $m = 1$. Player 1's strategy is to reduce the weight of every even vertex by 1 and move to the next odd vertex, eventually x. Player 1 starts the game by reducing $w_{0,0}$ by 1 and moving to 1. Player 2 reduces $w_{1,0}$ and moves to either 0 or 2. If Player 2 moves to 0, Player 1 can move back to 1 since $w_0 \geq w_1$. Player 2 will be forced to move to 2 to avoid losing by reasoning similar to that in Lemma 3.1. Play continues as above and Player 1 will move to x since x is odd. Player 2 reduces $w_{x,0}$ to 0 and either moves to $x-1$ or $x+1$. Either way, Player 1 will reduce the weight of w_{x-1} or w_{x+1} by 1 and move to x. Since $w_x = 0$, Player 1 wins.

For the induction step, assume Player 1 has the winning strategy for $1 \leq m$. Consider $w_x = m + 1$. The game begins as described above. Play continues and Player 1 will move to x since x is odd. Player 2 reduces $w_{x,0}$ such that $w_x \leq w_{x,0} - 1 = m + 1 - 1 = m$ and either moves to $x - 1$ or $x + 1$. Define a new game where the identity vertex is either $x - 1$ or $x + 1$ depending on where Player 2 moves. Then we have a game where $w_x = m$. Player 1 will reduce the weight of the identity in the new game w_{x-1} or w_{x+1} by 1 and move to x. Then Player 1 wins by the induction hypothesis. $\square$

Theorem 4.2. *Let G be the Cayley graph of any group generated by two involutions where $|G| = n$. Let x be the least such that $w_x \leq w_{x+1}$ for $x \in G$. If x is even, look for the greatest vertex $y \in G$ such that $w_y \leq w_{y-1}$. If y is odd, Player 1 wins. If y is even, Player 2 wins.*

Proof. Suppose y is odd. The proof is similar to Theorem 4.1. Instead of moving from 0 to 1, Player 1 will move from 0 to $n - 1$ and proceed in this direction. Then Player 1 wins by previous argument.

Suppose y is even. Player 1 starts the game by reducing w_0 and moving to either 1 or $n - 1$. If Player 1 moves to 1, Player 2 starts a new game on 1 as the identity. Then x is odd in relation to the identity in the new game, and Player 2 will take the role of Player 1 to move to x. Then Player 2 wins by Theorem 4.1. If Player 1 moves to $n - 1$, Player 2 starts a new game on $n - 1$ as the identity. Then y is odd in relation to the identity of the new game, and Player 2 will take the role of Player 1 to move to $n - 2$. Then Player 2 will reach y and win according to the above strategy. $\square$

Then we have winning strategies for all Cayley graphs of groups generated by two involutions.

Chapter 5

Quaternions

5.1 Introduction

We now consider another non-abelian group, the Quaternions. From [9] we have the following definition.

Definition 5.1. *Let*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where $i^2 = -1$. Then the quaternion group is of order 8 and we write $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Notice $|Q_8| = |D_4| = 8$; however these two groups have different structures. Consider the generating set $S = \{i, j\}$. Then we obtain the relations

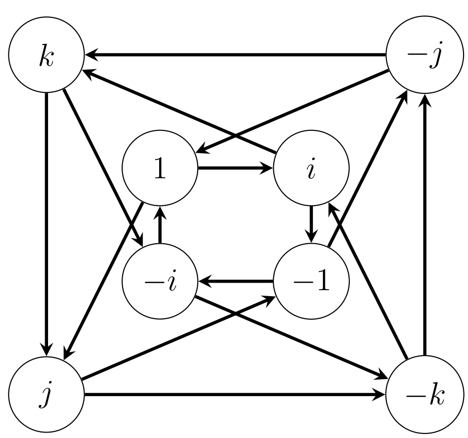
$$ij = k \qquad ji = -k$$

$$jk = i \qquad kj = -i$$

$$ki = j \qquad ik = -j$$

Now we are able to construct a Cayley graph of the Quaternions.

Example 12. *The Cayley graph of Q_8 generated by $\{i, j\}$.*



5.2 Winning Strategy

We begin to define strategy for Nim on Q_8 with the following Lemma.

Lemma 5.1. *Let the Player move to a vertex $x \in Q_8$. Then the Player can move to x as long as the game continues.*

Proof. Let G be the Cayley graph of Q_8. Suppose the Player moves to $x \in G$. Then the Opponent can either move to xi or xj, since $S = \{i, j\}$.

Case 1: Suppose the Opponent moves to xi. Then the Player moves to $xii = x(-1) = -x$. Again the Opponent can either move to $-xi$ or $-xj$. If the Opponent moves to $-xi$, the Player moves to $-xii = -x(-1) = x$ and we are done. If the Opponent moves to $-xj$, the Player moves to $-xjj = -x(-1) = x$. Then we conclude whenever the Opponent moves to xi, the Player will always be able to move back to x.

Case 2: Suppose the Opponent moves to xj. Then the Player can return to x by similar reasoning.

□

Lemma 5.1 is important because it allows a player the choice to return to that same vertex as long as the game continues. The other variable that will determine the winning strategy is the weight of the vertices. We define the minimum weight of Q_8 as the following.

Definition 5.2. *Define the minimum set M as the set of all vertices in Q_8 with the minimum initial weight m. Let m be the minimum of $\{w_{\pm 1}, w_{\pm i}, w_{\pm j}, w_{\pm k}\}$. Then $M = \{x \in Q_8 : w_x = m\}$.*

Using this definition, we see how a player wins by moving to some $x \in M$.

Lemma 5.2. *Let the Player be the first to move to a vertex $x \in M$. Then the Player has the winning strategy.*

Proof. We will use proof by induction. Suppose the Player moves to $x \in M$. Then $w_x = m$. For the initial case, let $m = 1$. The Opponent must reduce $w_{x,0}$ to 0 and move along an allowable edge. Note the Player always reduces the weight of the vertex by 1 to keep $w_x \leq w_y$ for all $y \in Q_8$. By Lemma 5.1, the Player will move to x after a cycle of play. Since $w_{y,0} > 0$ for all $y \in Q_8$, the game will continue until the Player moves back to x. Then the Opponent loses because $w_x = 0$.

For the induction step, suppose the Player has the winning strategy for $1 \leq m \leq n$. Consider $w_{x,0} = n+1$. The Opponent must reduce $w_{x,0}$ such that $w_x \leq w_{x,0} - 1 = n+1-1 = n$. By Lemma 5.1, the Player will move to x after a cycle of play since the Player always reduced by 1 to keep $w_x \leq w_y$ for all $y \in Q_8$. Then Player wins at x by the induction hypothesis. Note if the Opponent happened to reduce $w_{y,0}$ such that $w_y < w_{x,0} - 1 = n+1 = n$ for some $y \in Q_8$, then the Player wins on y by the induction hypothesis.

□

We refer to the above strategy as the Copycat Strategy. The Player will always be able to move to a vertex x as long as the Player moves along the edge with the same generator as the edge the Opponent moves along. According to the Copycat Strategy, if the Opponent moves along the i generator, the Player moves along the i generator. Similarly if the Opponent moves along the j generator, the Player moves along the j generator. We now apply this technique to generalize all winning strategies for Nim on Q_8.

These are the list of cases that will need to be proved. The number of the list corresponds with the Theorem number.

1. $1 \in M$
2. $i \in M$ or $j \in M$ [and $1 \notin M$]
3. $-1 \in M$ [and $1, i, j \notin M$]
4. k and $-k \in M$ [and $1, i, j, -1 \notin M$]
5. $M = \{k\}$ or $M = \{-k\}$
6. $M = \{-i\}$ or $M = \{-j\}$ or $M = \{-i, -j\}$
7. $M = \{k, -i\}$ or $M = \{-k, -j\}$
8. $M = \{k, -j\}$ or $M = \{-k, -i\}$
9. $M = \{k, -i, -j\}$ or $M = \{-k, -i, -j\}$

Theorem 5.1. *Let G be the Cayley graph of Q_8. If $1 \in M$, the Player 2 has the winning strategy.*

Proof. Suppose $1 \in M$. Then $w_1 \leq w_x$ for all $x \in Q$. At the start of the game Player 1 reduces $w_{1,0}$ such that $w_1 \leq w_{1,0} - 1$. Player 1 starting the game at 1 is equivalent to Player 2 moving to 1 to start the game. Thus we apply Lemma 5.2 to show Player 2 has the winning strategy. □

Theorem 5.2. *Let G be the Cayley graph of Q_8. If $i \in M$ or $j \in M$, then Player 1 has the winning strategy as long as $1 \notin M$.*

Proof. Suppose $i \in M$ and $1 \notin M$. Then $w_i < w_1$. Player 1 reduces w_1 by 1 and moves to i. Since $w_i \leq w_x$ for all $x \in Q_8$, Player 1 wins by Lemma 5.2.

Now suppose $j \in M$. Then Player 1 has the winning strategy by similar argument. □

Theorem 5.3. *Let G be the Cayley graph of Q_8. Suppose $1, i, j \notin M$. If $-1 \in M$, then Player 2 has the winning strategy.*

Proof. Player 1 starts the game by reducing w_1. Player 1 can either move to i or j. Player 2 reduces either w_i or w_j by 1 and moves to -1. Then Player 2 wins by Lemma 5.2. □

Theorem 5.4. *Let G be the Cayley graph of Q_8. If $k, -k \in M$ with $1, i, j, -1 \notin M$, then Player 2 has the winning strategy.*

Proof. Suppose $k, -k \in M$. Player 1 starts the game by reducing w_1 by some amount and moving to either i or j. If Player 1 moves to i, then Player 2 moves to $k \in M$ and wins by Lemma 5.2. If Player 1 moves to j, then Player 2 moves to $-k \in M$ and wins by Lemma 5.2. □

Theorem 5.5. *Let G be the Cayley graph of Q_8. If $M = \{k\}$ or $M = \{-k\}$, then Player 2 has the winning strategy.*

Proof. Suppose $k \in M$. Player 1 starts the game by reducing w_1 by some amount and moving to i or j. If Player 1 moves to i, Player 2 reduces w_i by 1 and moves to $k \in M$. Then Player 2 wins by Lemma 5.2.

If Player 1 moves to j, then Player 2 reduces w_j by 1 and moves to $-k$. Player 1 reduces w_{-k} by some amount and moves to either $-j$ or i. Regardless, Player 1 moves to $k \in M$ and wins by Lemma 5.2.

Now suppose $-k \in M$. Then Player 2 wins by similar argument. □

Theorem 5.6. *Let G be the Cayley graph of Q_8. If $M = \{-i\}$, $M = \{-j\}$, or $M = \{-i, -j\}$, then Player 1 has the winning strategy.*

Proof. Suppose $M = \{-i\}$. Player 1 reduces w_1 by 1 and moves to i. Player 2 either moves to k or -1. Either way, Player 1 can move to $-i \in M$. Then Player 1 wins by Lemma 5.2.

Suppose $M = \{-j\}$. Player 1 reduces w_1 by 1 and moves to j. Player 2 either moves to $-k$ or -1. Either way, Player 1 can move to $-j \in M$. Then Player 1 wins by Lemma 5.2.

Suppose $M = \{-i, -j\}$. Player 1 wins by similar reasoning. □

Theorem 5.7. *Let G be the Cayley graph of Q_8. If $M = \{k, -i\}$, then Player 2 has the winning strategy. If $M = \{-k, -j\}$, then Player 2 has the winning strategy.*

Proof. Suppose k and $-i$ are the only 2 elements in M. Player 1 starts the game by reducing w_1 and moving to either i or j. If Player 1 moves to i, then Player 2 moves to $k \in M$ and wins by Lemma 5.2.

If Player 1 moves to j, then Player 2 reduces w_j by 1 and moves to $-k$. Now Player 1 can move to either $-j$ or i. Either way, Player 2 can move to $k \in M$ and wins by Lemma 5.2.

Suppose $-k$ and $-j$ are the only 2 elements in M. Then Player 2 has the winning strategy by similar reasoning. □

Theorem 5.8. *Let G be the Cayley graph of Q_8. If $M = \{k, -j\}$, then Player 1 has the winning strategy. If $M = \{-k, -i\}$, then Player 1 has the winning strategy.*

Proof. Suppose k and $-j$ are the only 2 elements in M. Player 1 starts by reducing w_1 by 1 and moving to j. Player 2 can either move to -1 or $-k$. Either way Player 1 can move to $-j \in M$ and win by Lemma 5.2.

Suppose $-k$ and $-i$ are the only 2 elements in M. Then Player 1 has the winning strategy by similar reasoning. □

Theorem 5.9. *Let G be the Cayley graph of Q_8. If $M = \{k, -j, -i\}$, then Player 1 has the winning strategy. Similarly if $M = \{-k, -j, -i\}$, then Player 1 has the winning strategy.*

Proof. Suppose $k, -i, -j \in M$. Player 1 reduces w_1 by 1 and moves to j. Then Player 2 can either move to $-k$ or -1. Either way, Player 1 can move to $-j \in M$ and wins by Lemma 5.2.

Suppose $-k, -i, -j \in M$. Player 1 reduces w_1 by 1 and moves to i. Then Player 2 can either move to k or -1. Either way, Player 2 can move to $-i \in M$. Then Player 2 wins by Lemma 5.2. $\square$

These Theorems account for all weight distributions for Nim on Q_8. The following table nicely summarizes the results whenever $1, i, j, -1 \in M$.

Table 5.1: Nim on Q_8 Results Part 1

$x \in M$	$x \notin M$	Winner	Reference Theorem
1		P2	5.1
i or j	1	P1	5.2
-1	1, i, j	P2	5.3

Thus all the cases where $1, i, j$, or $-1 \in M$ are accounted for. Now we need to consider when M is a combination of $\pm k, -i, -j$. If $|M| = 1$, there are exactly four possible cases of M:

$$\{k\}, \{-k\}, \{-i\}, \{-j\}$$

If $|M| = 2$, there are exactly six possible cases of M:

$$\{k, -k\}, \{k, -i\}, \{k, -j\}, \{-k, -i\}, \{-k, -j\}, \{-i, -j\}$$

If $|M| = 3$, there are four possible cases of M:

$$\{k, -k, -i\}, \{k, -k, -j\}, \{k, -i, -j\}, \{-k, -i, -j\}$$

Finally if $|M| = 4$ there is only one case of M:

$$\{k, -k, -i, -j\}$$

If $|M| > 5$, then there must be some $x \in M$ such that $x \in \{1, -1, i, j\}$, and we have covered those cases in the above chart. Thus there are 15 cases, which are summarized in the following table.

Table 5.2: Nim on Q_8 Results Part 2

M	Winner	Reference Theorem
{k}	P2	5.5
{-k}	P2	5.5
{-i }	P1	5.6
{-j}	P1	5.6
{-i,-j}	P1	5.6
{k, -i}	P2	5.7
{-k,-j}	P2	5.7
{k, -j}	P1	5.8
{-k,-i}	P1	5.8
{k,-k}	P2	5.4
{k,-k,-i}	P2	5.4
{k,-k,-j}	P2	5.4
{k, -j, -i}	P1	5.9
{-k,-j,-i}	P1	5.9
{k,-k,-i,-j}	P2	5.4

Chapter 6

Abelian Groups of the form $\mathbb{Z}_2 \times \mathbb{Z}_n$

6.1 Introduction

We now direct our attention back to abelian groups. Recall a group G is abelian if for any $a, b \in G$, $ab = ba$. The structure of the abelian groups we are interested in follow from the Fundamental Theorem of Finite Abelian Groups.

Theorem 6.1. *Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups of the form*

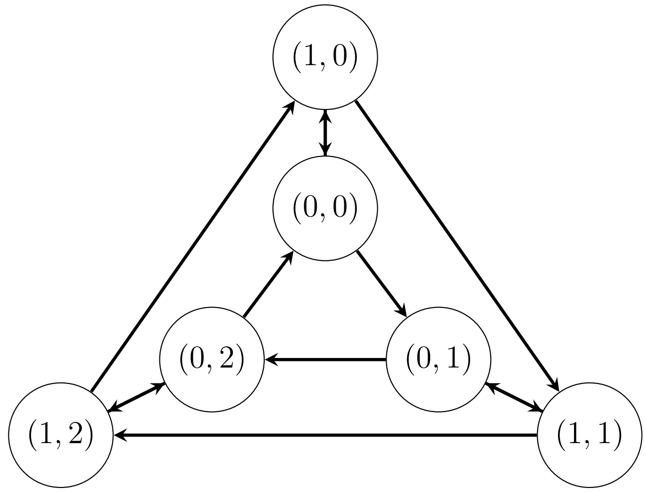
$$\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times ... \times \mathbb{Z}_{p_n^{\alpha_n}}$$

where the p_i's are primes, but not necessarily distinct.

We will be looking at Nim on the specific group, $\mathbb{Z}_2 \times \mathbb{Z}_n$. Let G be the Cayley graph of the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ for $2 < n \in \mathbb{N}$ and generating set $S = \{(1, 0), (0, 1)\}$. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a group generated by 2 involutions, which is covered in Chapter 4. Then G has $|\mathbb{Z}_2 \times \mathbb{Z}_n| = 2n$ vertices, with 2 edges initiating from each vertex giving G a total of $4n$ edges. Let $i, j \in \mathbb{Z}$. The vertices are labeled (i, j) for $0 \leq i \leq 1$ and $0 \leq j \leq n$. We define a vertex (i, j) to be *odd* if $i + j$ is odd and *even* if $i + j$ is even.

We call the cycle formed by $(0, j)$ the *Zero cycle* and the cycle formed by $(1, j)$ the *One cycle*. We say a player *switches cycles* if the player moves from $(0, j)$ to $(1, j)$ or vice versa. Each cycle is graph isomorphic to $\mathbb{Z}_n$ and contains no overlap since $(0, k)$ never equals $(1, m)$ for $k, m \in \mathbb{Z}$. These cycles account for $2n$ of the edges. The other $2n$ edges connect $(0, j)$ to $(1, j)$. Since $(0, j) + (1, 0) = (1, j)$ and $(1, j) + (1, 0) = (0, j)$, these are double sided arrows. Finally since $(0, j) + (0, 1) = (0, j + 1)$ and $(1, j) + (0, 1) = (1, j + 1)$, the Zero and One cycle have the same orientation, unlike the Cayley graphs of dihedral groups we saw earlier.

Example 13. *The Cayley graph for $\mathbb{Z}_2 \times \mathbb{Z}_3$ with $S = \{(1,0),(0,1)\}$ is below.*



6.2 Winning Strategy

Now that we have our gameboard, we start with playing techniques for Nim on $\mathbb{Z}_2 \times \mathbb{Z}_n$.

Lemma 6.1. *Let $j \in \{0, 1, ..., n-1\}$ be even. Then Player 1 can move to either $(0, j+1)$ or $(1, j)$ using only $(0, 1)$ and neither player hits a vertex twice.*

Proof. Let $(0, j)$ be a minimum counterexample. Then Player 1 can move to either $(0, j-1)$ or $(1, j-2)$ using only $(0, 1)$ without repeating vertices. Suppose Player 1 moves to $(0, j-1)$. If Player 2 moves to $(1, j-1)$, then Player 1 moves to $(1, j)$ and we have a contradiction. So Player 2 moves to $(0, j)$. Then Player 1 moves to $(0, j+1)$ and we have a contradiction again. Then we have shown if Player 1 moves to $(0, j-1)$, then Player 1 can move to either $(0, j+1)$ or $(1, j)$, which contradicts the fact that $(0, j)$ is a counterexample.

Suppose Player 1 moves to $(1, j-2)$. If Player 2 moves to $(1, j-1)$, then Player 1 moves to $(1, j)$ and we have a contradiction. So Player 2 must move to $(0, j-2)$. Then Player 1 moves to $(0, j-1)$. But since this is a counterexample, it is impossible to get to $(0, j-1)$ using only $(0, 1)$ generator with no repeats. Then either Player 1 used $(1, 0)$ or there was a repeat. Since Player 1 only uses $(0, 1)$, it must have been a repeat. Note there was not a repeat when Player 1 moved from $(1, j-3)$ to $(1, j-2)$. Then Player 2 moved to $(0, j-2)$ and Player 1 moved to $(0, j-1)$. This means $(1, j-2)$ or $(0, j-2)$ must have been used in paths to $(1, j-3)$. But the second component is non-decreasing until we get back to n, so we have a contradiction. Then this is not a counterexample.

□

Lemma 6.2. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$. Player 1 can move to any odd vertex $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n$ with each vertex being used no more than once. Similarly, Player 2 can move to any even vertex $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n$ with each vertex being used no more than once.*

Proof. Let $(0, j)$ be odd. By Lemma 6.1, Player 1 moves to either $(0, j-2)$ or $(1, j-1)$ only using the $(0, 1)$ generator with no repeat vertices.

Case 1: Player 1 moves to $(0, j-2)$. If Player 2 moves to $(0, j-1)$, then Player 1 moves to $(0, j)$ with no repeats since the second component is nondecreasing and we are done. If Player 2 moves to $(1, j-2)$, then Player 1 moves to $(1, j-1)$. If Player 2 moves to $(0, j-1)$, Player 1 moves to $(0, j)$. If Player 2 moves to $(1, j)$, then Player 1 moves to $(0, j)$. There are no repeats up until $(0, j-2)$ by Lemma 6.1. Notice in any of the above paths, each vertex gets used at most one. Then we are done with this case.

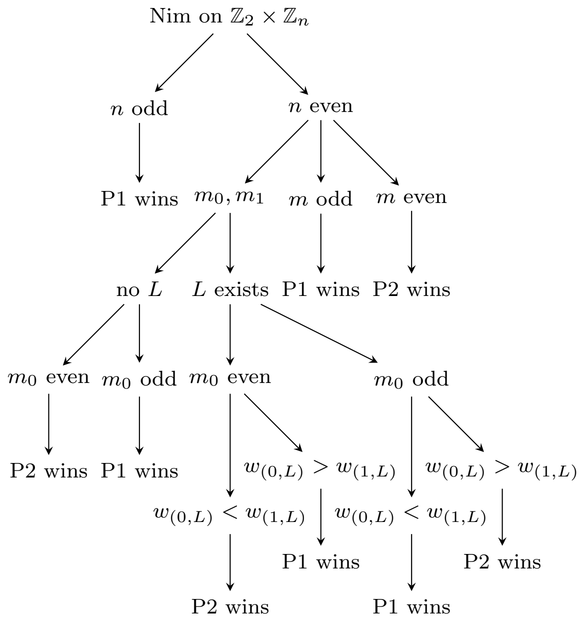
Case 2: Player 1 moves to $(1, j-1)$. If Player 2 moves to $(0, j-1)$, then Player 1 moves to $(0, j)$ and we are done since no vertex was used twice. If Player 2 moves to $(1, j)$, then Player 1 moves to $(0, j)$ and we are done. Note again there are no repeating vertices since the second component is nondecreasing.

Let $(1, j)$ be odd. Then Player 1 can move to $(1, j)$ by above reasoning.

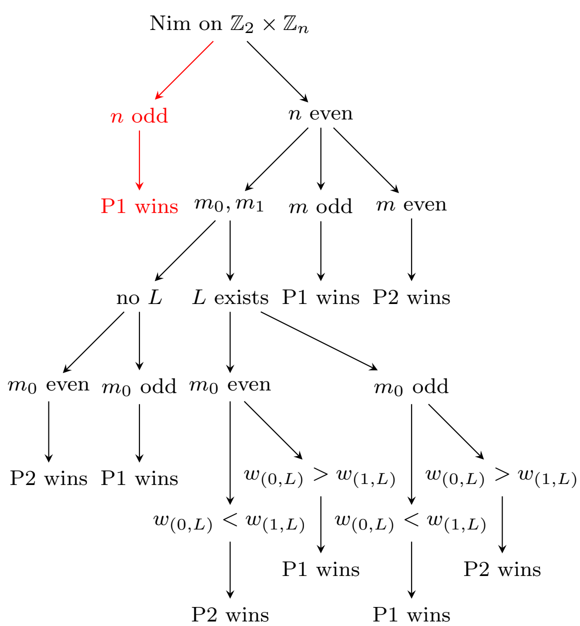
Similarly, if (i, j) is an even vertex, then Player 2 can move to (i, j) using each vertex at most once. □

Lemma 6.2 lets Player 1 move to any odd vertex and Player 2 move to any even vertex with each vertex along the path being used no more than one time. The strategy behind the lemma is that the Player who wishes to move to a specific vertex (i, j) can always move along the generator $(0, 1)$ to guarantee that no vertex gets hit twice on the way to any even or odd vertex. The only time the Player will move using the generator $(1, 0)$ is if the Player is trying to ultimately get to $(1, 0)$ or if the Opponent moves to $(i-1, j)$, which lets the Player to move to $(i-1, j) + (1, 0) = (i, j)$.

An important point to note in Lemma 6.2 is that every vertex along the way gets hit at most once. In terms of weight, this means if Player 1 is trying to get to some odd vertex (i, j), where $w_{(i,j)} < w_{(k,l)}$ for all odd vertices (k, l) where $k < i$, $l < j$, then once Player 1 moves to (i, j) we still have that $w_{(i,j)} \leq w_{(k,l)} - 1$, assuming that Player 1 always reduces the vertex weight by 1. This strategy will be illustrated in the following winning strategies for Nim on $\mathbb{Z}_2 \times \mathbb{Z}_n$. Here are the results.



We will prove each part of this diagram, as we did for Nim on D_n.



Theorem 6.2. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is odd. Then Player 1 has the winning strategy.*

Proof. Suppose n is odd. Player 1 starts by reducing $w_{(0,0)}$ to 0 and moving to $(0, 1)$. Player 1 can continue to move along the generator $(0, 1)$ no matter how Player 2 moves. Using the strategy outlined in Lemma 6.2, Player 1 can move to $(0, n)$ since n is odd without using any vertex more than once. Since all initial weights must be greater than 0 and each vertex could only have been reduced at most once, then no player loses before Player 1 moves to $(0, n)$. Notice $(0, n) = (0, 0)$. Since $w_{(0,0)} = 0$, Player 2 is unable to reduce $w_{(0,0)}$ and Player 1 wins.

□

The case where n is even does not simplify as easily. The following definition defines the minimum vertex on the Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is even and $n > 2$.

Definition 6.1. *Let w to be the minimum of the set*

$$\{w_{(0,0)}, w_{(0,1)}, ..., w_{(0,n-1)}, w_{(1,0)}, ..., w_{(1,n-1)}\}.$$

Then w is the minimum weight of G. Let

$$h_0 = \min\{j : w_{(0,j)} = w\}$$

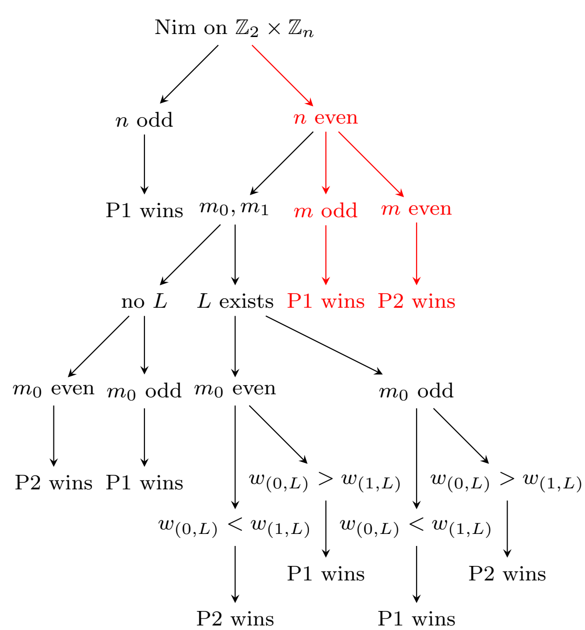
$$h_1 = \min\{j : w_{(1,j)} = w\}$$

Define

$$m = \begin{cases} (0, h_0) & \text{if } h_0 < h_1 \\ (1, h_1) & \text{if } h_0 > h_1. \end{cases}$$

If $h_0 = \emptyset$, then $m = (1, h_1)$. Similarly if $h_1 = \emptyset$, then $m = (0, h_0)$. We will refer to m as the minimum vertex. If $h_0 = h_1$, we have 2 minimum vertices where $m_0 = (0, h)$ and $m_1 = (1, h)$ for $h = h_0 = h_1$. We refer to m_0 as the Zero Cycle minimum and m_1 as the One Cycle minimum.

Then the *minimum vertex* $m = (i, j)$ is the vertex that has the minimum weight and is the least distance from the identity in terms of j. Theorem 6.3 addresses the case where there is only one minimum vertex m. Theorems 6.4, 6.5, and 6.6 address the case where there are two minimum vertices, m_0 and m_1.



Theorem 6.3. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is even and $n > 2$. Let $m \in G$ be the unique minimum vertex. If m is odd, Player 1 has the winning strategy. If m is even, Player 2 has the winning strategy.*

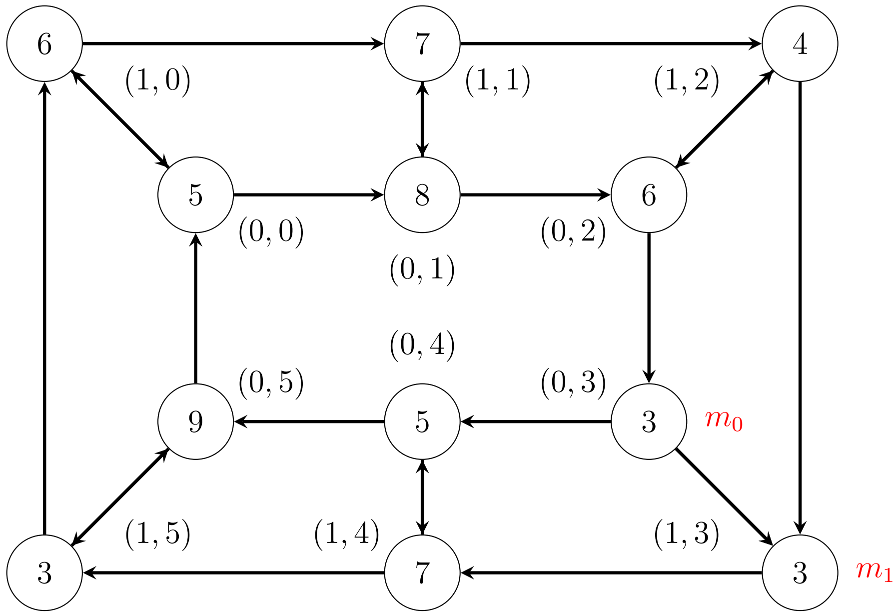
Proof. Suppose $m = (i, j)$ is odd. We will induct on w_m. For the initial case, suppose $w_m = 1$. Player 1's strategy is to reduce the weight of each vertex by 1 and move along the $(0, 1)$ generator. Since m is odd, Lemma 6.2 states Player 1 can move to m using each vertex along the way no more than one time. Then Player 2 must reduce the w_m to 0 and move to either $(i + 1, j)$ or $(i, j + 1)$. Consider the new game in which $(i + 1, j)$ or $(i, j + 1)$ is the starting vertex. In this new game, we define $m = (i, j) = (i, j + n)$. Since n is even, m remains odd in this new game. Then Player 1 moves to m by Lemma 6.2. If Player 2 reduced $w_{(k,l)}$ to 0 for any odd vertex (k, l) where $k < i, l < j$, then Player 2 could lose before Player 1 reaches m. Notice that for every even vertex (x, y), $w_{(x,y)} \geq w_{(x,y),0} - 1 \geq 1$, which means Player 1 will not lose on any even vertex (x, y) because Lemma 6.2 guarantees every vertex will be reduced no more than one time. Since $w_m = 0$, Player 2 is unable to reduce w_m and Player 1 wins.

For the induction step, assume Player 1 has the winning strategy for $1 \leq w_m \leq k$. Consider $w_m = k + 1$. Again Player 1's strategy is to reduce the weight of each even vertex by 1 and move along the $(0, 1)$ generator. Since m is odd, Player 1 moves to m by Lemma 6.2. We know no player loses before Player 1 moves to m since $w_{(x,y)} < w_m$ for all $x < i, y < j$. Then Player 2 reduces w_m such that $w_m \leq w_{m,0} - 1 = (k + 1) - 1 = k$ and moves to either $(i+1, j)$ or $(i, j+1)$. Consider the new game in which either $(i+1, j)$ or $(i, j+1)$ becomes the starting vertex. Then $m = (i, j) = (i, j + n)$ where $w_m \leq k$ in this new game unless Player 2 previously reduced a vertex x such that $w_x < w_m$. Either way we have that $w_x \leq w_m$ for all x Player 1 moves to. Then Player 1 wins by the Induction Hypothesis.

Suppose m is even. Then Player 2 wins by similar argument. □

Now we have the winning strategies for Nim on $\mathbb{Z}_2 \times \mathbb{Z}_n$ when there is only one minimum vertex. The case not covered is if $h_0 = h_1$. Basically in this situation, the two vertices with the minimum weight share the same $j \in \mathbb{Z}_n$ coordinate lined up, but are located in opposite cycles. If $(0, h_0)$ is odd, then Player 1 wants to keep play along the $(0, 1)$ generator. On the other hand, if $(1, h_1)$ is odd, then Player 1 wants to keep play along the One cycle also along the $(0, 1)$ generator. Once Player 1 moves to the minimum odd vertex before Player 2 can move to the corresponding even minimum vertex, then Player 1 is wins by Lemma 6.2. Once Player 1 moves to the minimum odd vertex, Lemma 6.2 states Player 1 can move back to that vertex using each vertex along the way only once. Then the determining factor of winning strategy on these games is the initial weights of the vertices before the minimum vertices.

Example 14. *The situation when $h_0 = h_1$ on $\mathbb{Z}_2 \times \mathbb{Z}_6$.*



In the above game we see $w = 3$, $m_0 = (0, 3)$ and $m_1 = (1, 3)$. Notice either m_0 is odd and m_1 is even or m_1 is odd and m_0 is even. Then the Zero Cycle minimum and the One Cycle minimum cannot be both even or both odd.

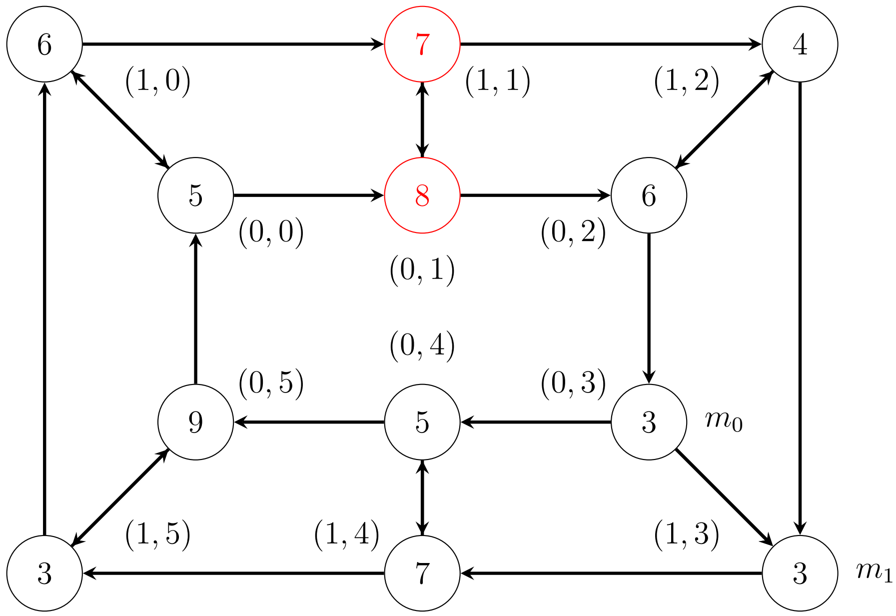
The following theorem defines the winning strategy on these types of games. First we need the following definition of L.

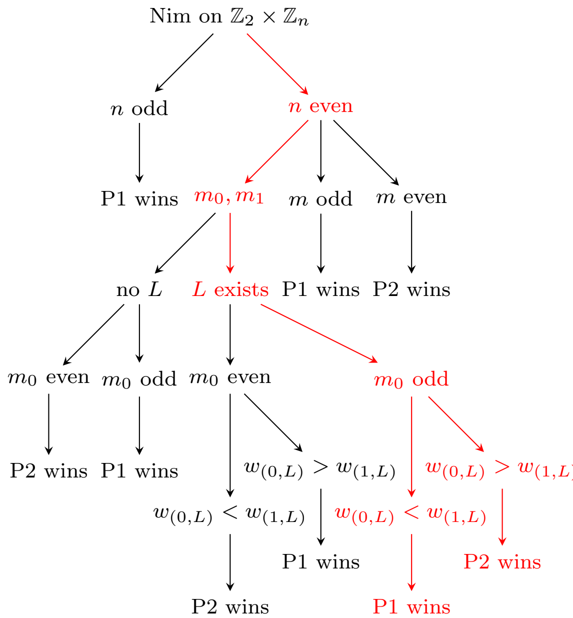
Definition 6.2. *Let L be the largest positive integer such that*

- $L < h$
- $L \equiv h \mod 2$ *for $h = h_0 = h_1$ as defined above*
- $w_{(0,L)} \neq w_{(1,L)}$

If L does not exist, then we can assume $w_{(0,k)} = w_{(1,k)}$ for all $k \equiv h \mod 2$.

Example 15. *In this game, L=1 since $w_{(0,L)} \neq w_{(1,L)}$. Player 2 has the winning strategy by Theorem 6.4*



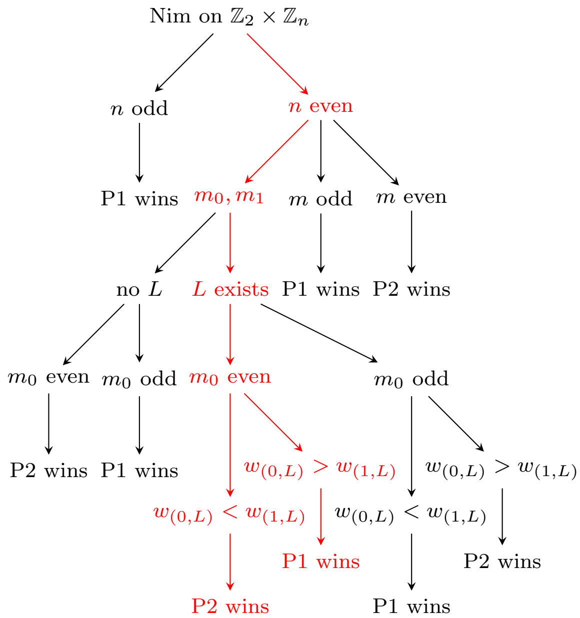


Theorem 6.4. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is even and $n > 2$. Let $h_0 = h_1 = h$. Assume L exists. Suppose m_0 is odd. Player 1 has the winning strategy if $w_{(0,L)} < w_{(1,L)}$. Player 2 has the winning strategy if $w_{(0,L)} > w_{(1,L)}$.*

Proof. Suppose $m_0 = (0, j)$ is odd and $w_{(0,L)} < w_{(1,L)}$. Player 1 starts by reducing $w_{(0,0),0}$ by 1. Player 1's strategy is to move along the $(0, 1)$ generator and eventually move to $(0, L)$ whether it be along the $(0, 1)$ or $(1, 0)$ generator. Since m_0 is odd, we know h is odd and thus $L \equiv h \mod 2$ is odd. Then Player 1 can move to $(0, L)$ by Lemma 6.2. Either Player 1 reduced $w_{(1,L)}$ or $w_{(0,L-1)}$ before moving to $(0, L)$. Notice if Player 1 reduced $w_{(1,L),0}$, then when Player 1 moves to $(0, L)$ we have $w_{(0,L),0} \leq w_{(1,L)}$. If Player 1 reduced $w_{(0,L-1),0}$, then we have $w_{(0,L),0} < w_{(1,L),0}$. Player 2 reduces $w_{(0,L)}$ and is forced to move along the Zero Cycle to $(0, L + 1)$ by Lemma 3.2. Player 1 continues to move along the Zero Cycle. Since $w_{(0,k),0} = w_{(1,k),0}$ for all odd $k < j$, Player 1 can force Player 2 to continue along the Zero Cycle by Lemma 3.2. Then Player 1 will move to m_0, and Player 1 wins by Theorem 6.3.

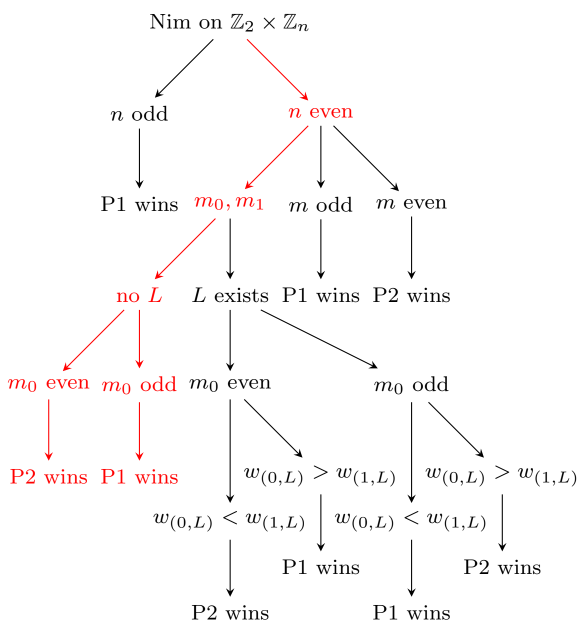
Suppose $w_{(0,L)} > w_{(1,L)}$. Since m_0 is odd, then $m_1 = (1, j)$ is even. Player 1 starts by reducing $w_{(0,0),0}$ and moving to either $(1, 0)$ or $(0, 1)$. Regardless, Player 2's strategy is to move along the $(0, 1)$ generator and eventually move to $(1, L)$. We know $(1, L)$ is even since $(0, L)$ is odd. Then Player 2 can move to $(1, L)$ by Lemma 6.2. Either Player 2 reduced $w_{(0,L)}$ or $w_{(1,L-1)}$ before moving to $(0, L)$. Notice if Player 2 reduced $w_{(0,L),0}$, then when Player 2 moves to $(1, L)$ we have $w_{(1,L),0} \leq w_{(0,L)}$. If Player 2 reduced $w_{(1,L-1),0}$, then we have $w_{(1,L),0} < w_{(0,L),0}$. Player 1 reduces $w_{(0,L)}$ and is forced to move along the One Cycle to $(1, L + 1)$ by Lemma 3.2. Player 2 continues to move along the One Cycle. Since $w_{(1,k),0} = w_{(0,k),0}$ for all odd $k < j$, Player 2 can force Player 1 to continue along the One Cycle by Lemma 3.2. Then Player 2 will move to m_1, and Player 2 wins by Theorem 6.3.

□



Theorem 6.5. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is even and $n > 2$. Let $h_0 = h_1 = h$. Assume L exists. Suppose m_0 is even. Player 2 has the winning strategy if $w_{(0,L)} < w_{(1,L)}$. Player 1 has the winning strategy if $w_{(0,L)} > w_{(1,L)}$.*

Proof. The proof is similar to the argument in Theorem 6.4. $\square$



Theorem 6.6. *Let G be the Cayley graph for the group $\mathbb{Z}_2 \times \mathbb{Z}_n$ where n is even and $n > 2$. Let $h_0 = h_1 = h$. Assume there is no L. Player 1 has the winning strategy if m_0 is odd, and Player 2 has the winning strategy if m_0 is even.*

Proof. Suppose $m_0 = (0, h)$ is odd. We will induct on w_{m_0}. For the initial case, suppose $w_{m_0} = 1$. Player 1's strategy is to reduce the weight of each vertex by 1 and move along the $(0, 1)$ generator. Since L does not exist, then $w_{(0,l),0} = w_{(1,l),0}$ for all $l \equiv h \mod 2$. To start the game, Player 1 reduces $w_{(0,0),0}$ by 1 and moves to $(0, 1)$. Player 2 reduces the weight of $(0, 1)$ such that $w_{(0,1)} \leq w_{(0,1),0} - 1 < w_{(1,1),0}$. Then Player 1 can force Player 2 to move to $(0, 2)$ by Lemma 3.2. Play continues until evenutally Player 1 moves to m_0. Player 2 must reduce w_{m_0} to 0 and move to either $(0, h + 1)$ or $(1, h)$. Player 1 can move along the $(0, 1)$ generator to either $(n, 1)$ or $(0, 1)$ by Lemma 6.2.

Suppose Player 1 moves to $(n, 1)$. Then Player 2 can either move to $(0, 0)$ or $(1, 1)$. If Player 2 moves to $(0, 0)$, then Player 1 moves to $(0, 1)$. Then since $w_{(0,l)} \leq w_{(1,l)}$ for all $l \equiv h \mod 2$, Player 1 can force Player 2 along the Zero Cycle by Lemma 3.2. If Player 2 moves to $(1, 1)$, then Player 1 reduces $w_{(1,1)}$ by 1 and moves to $(0, 1)$. Then $w_{(0,1)} \leq w_{(1,1)}$ and Player 2 is forced along the Zero Cycle by Lemma 3.2. Then Player 1 moves to m_0. Since $w_{m_0} = 0$, Player 1 wins.

Suppose Player 1 moves to $(0, 1)$. Then Player 1 wins by reasoning from above.

For the induction step, suppose Player 1 has the winning strategy for $1 \leq w_{m_0} \leq k$. Consider $w_{m_0} = k + 1$. Player 1's strategy is to reduce the weight of each vertex by 1 and move along the $(0, 1)$ generator. Since L does not exist, then $w_{(0,l),0} = w_{(1,l),0}$ for all odd $l < h$. To start the game, Player 1 reduces $w_{(0,0),0}$ by 1 and moves to $(0, 1)$. Player 1 forces Player 2 to move along the Zero Cycle by Lemma 3.2. Play continues until eventually Player 1 moves to m_0. Player 2 must reduce w_{m_0} such that $w_{m_0} \leq w_{m_0,0} - 1 = k + 1 - 1 = k$ and move to either $(0, h + 1)$ or $(1, h)$. Player 1 can move along the $(0, 1)$ generator to either $(n, 1)$ or $(0, 1)$ by Lemma 6.2.

Suppose Player 1 moves to $(n, 1)$. Then Player 2 can either move to $(0, 0)$ or $(1, 1)$. If Player 2 moves to $(0, 0)$, then Player 1 moves to $(0, 1)$. If Player 2 moves to $(1, 1)$, then Player 1 reduces $w_{(1,1)}$ by 1 and moves to $(0, 1)$. Notice $w_{(0,1)} \leq w_{(1,1)}$ since Player 1 forced Player 2 to reduce $w_{(0,1)}$ such that $w_{(0,1)} \leq w_{(1,1)}$ from previous round. Then we have a new game where $w_{m_0} \leq k$. Then Player 1 wins by the Induction Hypothesis.

Suppose Player 1 moves to $(0, 1)$. Then Player 1 wins by above argument.

Suppose m_0 is even. Then Player 2 wins by similar induction. □

Then we have the Winning Strategy for all possible cases of Nim on $\mathbb{Z}_2 \times \mathbb{Z}_n$. To summarize, if n is odd, Player 1 has the winning strategy. If n is even, we look at the vertex with the minimum weight, m. If m is odd, Player 1 has the winning strategy. If m is even, Player 2 has the winning strategy.

Chapter 7

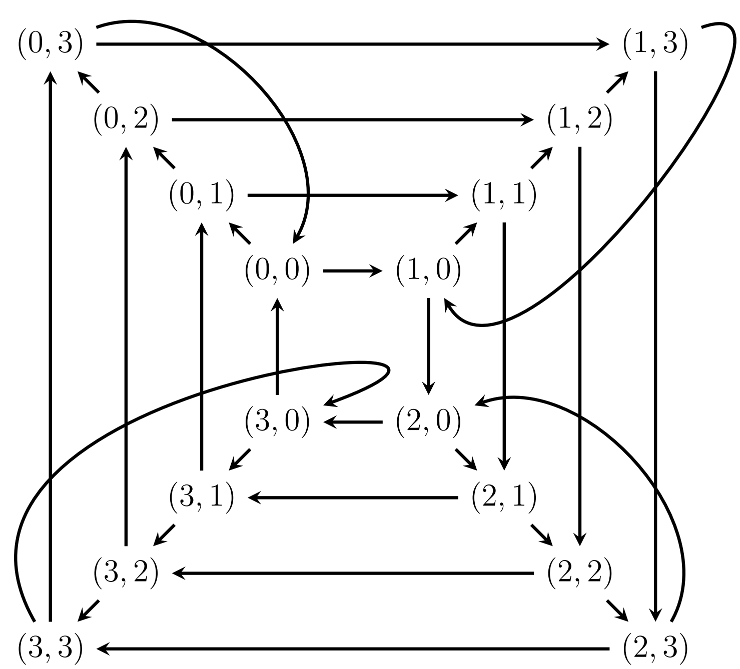
Open Questions

7.1 $\mathbb{Z}_n \times \mathbb{Z}_n$ Partial Solution

We next seek a general solution to Nim on $\mathbb{Z}_n \times \mathbb{Z}_n$. However, we are not yet able to completely determine winning strategy for all initial weight distributions. We are able, however, to define the rules of the game and provide winning strategies for select cases.

Let G be the Cayley graph of the group $\mathbb{Z}_n \times \mathbb{Z}_n$ where n is even and $n \in \mathbb{N}$. Then G has n^2 vertices corresponding to the n^2 elements in $\mathbb{Z}_n \times \mathbb{Z}_n$, and $2n^2$ edges because each vertex has 2 edges initiating from it. $\mathbb{Z}_n \times \mathbb{Z}_n$ consists of n cycles of $\mathbb{Z}_n$. The cycles have the same orientation. Exactly n^2 of the edges connect (i, j) to $(i, j + 1)$. The other n^2 edges connect (i, j) to $(i + 1, j)$.

Example 16. *The Cayley graph for $\mathbb{Z}_4 \times \mathbb{Z}_4$ is below.*



It will be convenient to use cosets defined in [9] to define game play techniques.

Definition 7.1. *Let G be a group and H be a subgroup of G. Define a coset of H with $g \in G$ to be the set*

$$g + H = \{g + h : h \in H\}.$$

For $G = \mathbb{Z}_n \times \mathbb{Z}_n$, we define the set

$$H = \langle(1,1)\rangle = \{(0,0),(1,1),(2,2),...,(n-1,n-1)\}$$

to create our cosets. The game is divided into n cosets, $(i,0)+H$ for each $i \in \{0,1,...,n-1\}$. Here are some examples of cosets:

$$(0,0) + H = \{(0,0),(1,1),(2,2),...,(n-1,n-1)\}$$

$$(1,0) + H = \{(1,0),(2,1),(3,2),...,(0,n-1)\}$$

$$(2,0) + H = \{(2,0),(3,1),(4,2,),...,(1,n-1)\}$$

$$...$$

$$(n-1,0) + H = \{(n-1.0),(0,1),(1,2),(2,3),...(n-2,n-1)\}$$

We say $(i,0)+H$ is odd if i is odd, and $(i,0)+H$ is even if i is even. Define w_N to be the $\min\{w_{(x,y)} : (x,y) \in (N,0)+H\}$ for $N \in \mathbb{N}$. We call w_N the minimum weight of the coset $(N,0)+H$.

Notice when a player moves from any $(x,y) \in (N,0)+H$ along the $(1,0)$ generator, the player moves from $(x,y) = (N,0)+(i,i)$ to $(1,0)+[(N,0)+(i,i)] = (N+1,0)+(i,i)$. When a player moves along the $(0,1)$ generator, the player moves from (x,y) to $(0,1)+(x,y) = (-1,0)+(x+1,y+1) = (-1,0)+(1,1)+(N,0)+(i,i) = (N-1,0)+(i+1,i+1)$. Basically moving along the $(1,0)$ generator moves from the coset $(N,0)+H$ to $(N+1,0)+H$, while moving along the $(0,1)$ generator moves from the coset $(N,0)+H$ to $(N-1)+H$. The understanding of this concept is important in the following winning strategy.

In the following lemma, we use nonreduced integers in the ordered pairs. For example, $(0,0)$ becomes (n,n) after a player moves from it.

Lemma 7.1. *Let G be the Cayley graph of $\mathbb{Z}_n \times \mathbb{Z}_n$. Suppose the Player moves to $(N,0)+(h,h)$ for $h \leq n$. Let $w_N < w_{N+1}$ and $w_N < w_{N-1}$. Define $(N,0)+(i,i)$ to be the vertex with the minimum weight. That is, i is the least such that $w_{(N,0)+(i,i)} = w_N$ and $i > k$. Then the Player has the winning strategy.*

Proof. We will induct on w_N. For the initial case, suppose $w_N = 1$. For $h \leq g \leq i-1$, the Opponent reduces $w_{(N,0)+(g,g)}$ by some amount and moves either along the $(1,0)$ generator to $(N+1,0)+(g,g)$ or along the $(0,1)$ generator to $(N,1)+(g,g) = (N-1,0)+(g+1,g+1)$. If the Opponent moves to $(N+1,0)+(g,g)$, the Player reduces $w_{(N+1,0)+(g,g)}$ by 1 and moves to $(N,0)+(g+1,g+1)$. If the Opponent moves to $(N-1,0)+(g+1,g+1)$, the Player reduces $w_{(N-1,0)+(g+1,g+1)}$ by 1 and moves to $(N,0)+(g+1,g+1)$. The Player's strategy is to always reduce the weight of the vertex by 1 and move to a vertex in the coset $(N,0)+H$. Then gameplay is confined to vertices in the cosets $(N-1,0)+H, (N,0)+H$, and $(N+1,0)+H$. Since all vertices in these cosets have an initial weight greater than 0, play continues until the Player moves to $(N,0)+(i,i)$. The Opponent reduces $w_{(N,0)+(i,i)}$ to 0. For $i \leq h+n$, play continues as described above. We claim here that the Player moves to $(N,0)+(h,h)$. Note that since the Player only reduced each vertex by 1, $w_{(N+1,0)+(g+n,g+n)} = w_{(N+1,0)+(g,g),0} - 1 \geq 1$ and $w_{(N-1,0)+(g+n,g+n)} = w_{(N-1,0)+(g,g),0} - 1 \geq 1$ for $g \leq i$. Then the Player cannot lose before moving to $(N,0)+(h+n,h+n)$. If the Opponent reduces $w_{(N,0)+(g,g),0}$ to 0, then the Opponent loses at $(N,0)+(g+n,g+n)$. Otherwise the Opponent must reduce $w_{(N,0)+(i,i)}$. Since $w_{(N,0)+(i,i)} = 0$, the Player wins.

For the induction step, assume the Player has the winning strategy for $1 \leq w_{(N,0)+(i,i)} \leq k$. Consider $w_{(N,0)+(i,i)} = k+1$. For $h \leq g \leq i-1$, the Opponent reduces $w_{(N,0)+(g,g)}$ by some amount and moves to either $(N+1,0)+(g,g)$ or $(N-1,0)+(g+1,g+1)$. Either way, the Player reduces the weight by 1 and moves to $(N,0)+(g+1,g+1)$. Play continues until the Player moves to $(N,0)+(i,i)$. Then the Opponent reduces $w_{(N,0)+(i,i),0}$ such that $w_{(N,0)+(i,i)} \leq w_{(N,0)+(i,i),0} - 1 = (k+1) - 1 = k$. For $i \leq h$, play continues as described above. Once the Player moves to $(N,0)+(h+n,h+n)$, we have a new game. Note $w_{(N,0)+(g,g)} \leq w_{(N+1,0)+(g,g)}$ and $w_{(N,0)+(g,g)} \leq w_{(N-1,0)+(g+1,g+1)}$. Then the Player wins by the Induction Hypothesis. □

We see that if the minimum weight of a coset is strictly less than the minimum weight of the immediately preceding and following coset, then the Player who moves to that coset first wins. The Player wins by always moving to the coset with the minimum value whenever the Opponent moves to one of the surrounding cosets. The Player restricts play to those 3 cosets. Since one coset has a smaller minimum weight, the Player wins by constantly moving to that particular coset, which forces the Opponent to always reduce the minimum weight.

The following lemma addresses the case where the minimum weight of cosets are equal. The Player can still win as long as the Player moves to the vertex with the minimum coset weight before the Opponent moves to the vertex with that same weight in either the preceding or following coset. The inequalities at the end of the theorem basically say if the distance to the lowest weight on $(N,0)+H$ is less than both the distance to the lowest weights on $(N+1,0)+H$ and $(N-1,0)+H$, then the Player wins. The inequality concerning $(N,0)+H$ and $(N-1,0)+H$ is strictly less than because moving once along $(0,1)$ from a vertex $(N,0)+(h,h)$ gives us $(N-1,0)+(k,k)$ where $k=h+1$ while moving once along the $(1,0)$ generator gives us $(N+1,0)+(k,k)$ where $k=h$. Thus a player will move to $(N-1,0)+(h,h)$ before the other player can move to $(N,0)+(h,h)$, and hence the need for $i-h < d-h$.

Lemma 7.2. *Suppose the Player moves to $(N,0)+(h,h)$. Let $w_N \leq w_{N+1}$ and $w_N \leq w_{N-1}$. Define the vertices with the minimum weights as below.*

$$(N,0)+(i,i) \qquad i = \min\{j \in \{0,1,...,n-1\} : j \geq h, w_{(N,0)+(j,j)} = w_N\}$$

$$(N+1,0)+(c,c) \qquad c = \min\{j \in \{0,1,...,n-1\} : j \geq h, w_{(N+1,0)+(j,j)} = w_{N+1}\}$$

$$(N-1,0)+(d,d) \qquad d = \min\{j \in \{0,1,...,n-1\} : j \geq h, w_{(N-1,0)+(j,j)} = w_{N-1}\}$$

If $i-h \leq c-h$ and $i-h < d-h$, then the Player wins.

Proof. Suppose the Player moves to $(N,0)+(h,h)$. The Player's strategy is to always reduce the weight of the vertex by 1 and move to a vertex in the coset $(N,0)+H$. For $h \leq g \leq i-1$, the Opponent reduces $w_{(N,0)+(g,g)}$ by some amount and moves either along the $(1,0)$ generator to $(N+1,0)+(g,g)$ or along the $(0,1)$ generator to $(N,1)+(g,g) = (N-1,0)+(g+1,g+1)$. If the Opponent moves to $(N+1,0)+(g,g)$, the Player reduces $w_{(N+1,0)+(g,g)}$ by 1 and moves to $(N,0)+(g+1,g+1)$. If the Opponent moves to $(N-1,0)+(g+1,g+1)$, the Player reduces $w_{(N-1,0)+(g+1,g+1)}$ by 1 and moves to $(N,0)+(g+1,g+1)$. Since all vertices in $(N-1,0)+(g,g)$, $(N,0)+(g,g)$, and $(N+1,0)+(g,g)$ have weight greater than w_{N-1}, w_N, and w_{N+1}, respectively, play will continue in this way until the Player moves to $(N,0)+(i-1,i-1)$.

Then the Opponent moves to either $(N+1,0)+(i-1,i-1)$ or $(N-1,0)+(i,i)$. The Player will reduce either vertex by 1 and move to $(N,0)+(i,i)$. Notice that $w_{(N+1)+(c,c)}$ and $w_{(N-1)+(d,d)}$ have not been reduced yet since $d > i$ and $c > i-1$. Then the Opponent reduces $w_{(N,0)+(i,i)}$ such that $w_N < w_{N+1}$ and $w_N < w_{N-1}$. The Opponent then moves to either $(N+1)+(i,i)$ or $(N-1,0)+(i+1,i+1)$. Consider the new game in which $(N+1,0)+(i,i)$ or $(N-1,0)+(i+1,i+1)$ is the starting vertex. The Player starts this new game by reducing the starting vertex by 1 and moving to $(N,0)+(i+1,i+1)$. Since $w_N < w_{N+1}$ and $w_N < w_{N-1}$, then the Player has the winning strategy by Lemma 7.1. □

Using these game play techniques, we are able to define winning strategies on some of the initial weight conditions for Nim on $\mathbb{Z}_n \times \mathbb{Z}_n$.

Theorem 7.1. *Let G be the Cayley graph for the group $\mathbb{Z}_n \times \mathbb{Z}_n$. Suppose there exists a coset $(N,0)+H$ that fulfills the conditions from Lemma 7.2. If $N = 0$, Player 2 has the winning strategy.*

Proof. Suppose $N = 0$. Player 1 reduces $w_{(0,0)}$ by some amount and moves to either $(1,0) \in (1,0)+H$ or $(0,1) \in (n-1,0)+H$. Either way, Player 2 reduces the vertex by 1 and moves to $(1,1) \in (0,0)+H$. Then since $w_{(0,0)+H} \leq w_{(1,0)+H}$, $w_{(0,0)+H} \leq w_{(n-1,0)+H}$ and by the assumption that Player 2 moves to w_N before Player 1 moves to w_{N+1} or w_{N-1}, then Player 2 wins by Lemma 7.2. □

Theorem 7.2. *Let G be the Cayley graph for the group $\mathbb{Z}_n \times \mathbb{Z}_n$. Suppose there exists a coset $(N,0)+H$ that fulfills the conditions from Lemma 7.2. If $N = 1$ or $N = n-1$, Player 1 has the winning strategy.*

Proof. Suppose $N = 1$. Player 1 reduces $w_{(0,0)}$ by 1 and moves to $(1,0) \in (1,0)+H$. Then Player 1 wins by Lemma 7.2.

Suppose $N = n-1$. Player 1 reduces $w_{(0,0)}$ by 1 and moves to $(0,1) \in (n-1,0)+H$. Then Player 1 wins by Lemma 7.2. □

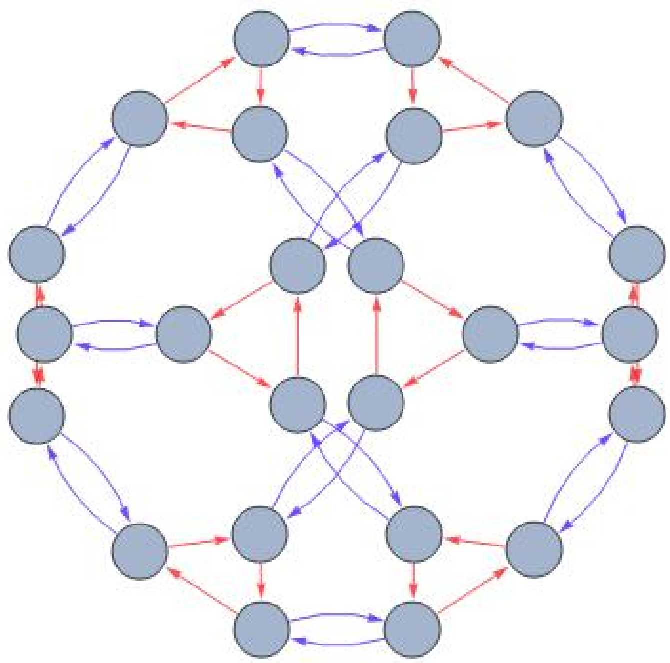
The general strategy so far for Nim on $\mathbb{Z}_n \times \mathbb{Z}_n$ is to check the cosets $(0,0)+H$, $(1,0)+H$, and $(n-1,0)+H$. If the criteria meets that of Theorem 7.1, the winning strategy is already determined. If this is not the case, a player should try to move to the minimum initial weight in order to trap their opponent. As of now, I have winning strategies for $\mathbb{Z}_n \times \mathbb{Z}_n$ for $n \leq 5$, but it is unclear how to generalize these results to games of more complexity.

7.2 Symmetric Groups Partial Solution

We also seek to complete winning strategies for Nim on the permutation groups S_n. We define a symmetric group as found in [9].

Definition 7.2. *The symmetric group on n letters, S_n, is a group with $n!$ elements, where the binary operation is the composition of maps.*

Notice we already have the winning strategies for Nim on S_3 since $S_3 \cong D_3$. We will present the complete solution to Nim on S_4 to provide possible insight to the general game on S_n. Let G be the Cayley graph of S_4 with generating set $S = \{(123), (34)\}$. We will refer to the respective generators as r, s where $r = (1, 2, 3)$ and $s = (3, 4)$. Then G has 24 vertices corresponding with the 24 elements of S_4. Each vertex has 2 edges initiating from it because $|S| = 2$. Notice the generator $(3, 4)$ is an involution, which means 24 of the edges will be used to create 12 double sided arrows. The other 24 edges connect x to $x(1, 2, 3)$ for all $x \in S_4$.



We define the *8-Triumph Cycle* similarly to the Triumph Cycle we used with Dihedral Groups. So far we have the winning strategy for S_4 with the generating set $S = \{(123), (34)\}$. I refrain from presenting results until it becomes clear how to generalize.

7.3 Future Research

We have introduced the game Nim on the Cayley graphs of groups by completely solving the game on Cyclic Groups, Dihedral Groups, finite groups of two involutions, the Quaternions, and abelian groups of the form $\mathbb{Z}_2 \times \mathbb{Z}_n$. We also have partial solutions to Nim on $\mathbb{Z}_n \times \mathbb{Z}_n$ and S_n. The immediate goal of research is to generalize solutions on these games. A further analysis of winning strategies would reveal which player has a better probability of winning the game with randomly chosen initial conditions. We could even investigate the number of moves it takes for a player to win a given game of Nim with a set of initial conditions and seek to minimize or maximize this number. There are certainly more groups to consider, particularly S_n, A_n, Dic_n, and $\mathbb{Z}_n \times \mathbb{Z}_m$. The results of Nim on these other groups have the potential to yield significant results.

We have considered one type of game on groups using a particular set of rules. There are many possible rule alterations to be made.

- Recall the rule that the initial weight of each vertex must be a finite number greater than 0. What happens if we do allow vertices to begin with weight 0? Then we would have to consider the strategy of the players to race to vertices with initial weight 0.
- What happens if we allow weights to be infinite? This would involve some further rule restrictions, such as players can only reduce the weight to 0 or leave the weight at infinity. Clearly this game is considerably different than the game we introduced, but it may yield interesting results.
- How would these games go if we placed the weight on the edges instead of the vertices?
- What happens if there are more than two players? What would Nim with three players entail?

The variations to Nim on groups are endless, and each has the potential to give us insight to further group properties.

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Nim on Groups

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