2008

# A classification of certain maximal subgroups of alternating groups

Bret Benesh
*College of Saint Benedict/Saint John's University*, bbenesh@csbsju.edu



## Recommended Citation

Benesh B. 2008. A classification of certain maximal subgroups of alternating groups. In: Kappe L-C, Magidin A, Morse RF, editors. *Contemporary Mathematics 470: Computational Group Theory and the Theory of Groups, AMS Special Session on Computational Group Theory, March 3-4, 2007, Davidson, North Carolina*. Providence (RI): American Mathematical Society. p. 21-26.

Contemporary Mathematics
Volume 470, 2008

# A Classification of Certain Maximal Subgroups of Alternating Groups

Bret Benesh

Abstract. This paper addresses an extension of Problem 12.82 of the Kourovka notebook, which asks for all ordered pairs $(n, m)$ such that the symmetric group $S_n$ embeds in $S_m$ as a maximal subgroup. Problem 12.82 was answered in a previous paper by the author and Benjamin Newton. In this paper, we will consider the extension problem where we allow either or both of the groups from the ordered pair to be an alternating group.

## 1. Introduction

While graduate students enrolled in a computational group theory course, the author and Benjamin Newton encountered problem 12.82 of the Kourovka Notebook [**5**]. This problem, submitted by V. I. Suschanskiĭ, poses the question of describing the set $\mathcal{M}$ of all pairs of positive integers $(n, m)$ such that the symmetric group $S_m$ contains a maximal subgroup isomorphic to $S_n$. One obvious family of such pairs is

$$\{(n, n+1) \mid n \geq 1\}.$$

The goal of the course was to provide an answer to this question with the help of the computational group theory system Magma [**1**]. A review of the literature indicated that a second family [**2, 3**] was known:

$$\left\{(n,m) \;\middle|\; m = \binom{n}{k},\ 2 \leq k \leq n/2 - 1,\ \binom{n-2}{k-1} \text{ is odd}\right\}.$$

Magma was used to check the maximal subgroups of symmetric groups of small degree, and it was determined that these two families did not constitute a complete solution to Suschanskiĭ's question. The data generated by Magma led to a discovery of a third family [**6**]:

$$\left\{\left(kr, \frac{(kr)!}{(r!)^k k!}\right) \;\middle|\; k, r > 1,\ k + r \geq 6,\ (k, r) \in \mathcal{C}\right\},$$

where $\mathcal{C}$ is defined to be the set of all ordered pairs of the form $(2, 2^d+1), (3, 2^e+1)$, or $(2l, 2)$ for $d \geq 0, e \geq 1$, and $l \geq 2$. It was proved in [**6**] that these are the only three possible families, and all such ordered pairs lie in one of the three families. In

2000 *Mathematics Subject Classification.* 20B35, 20E28.
*Key words and phrases.* symmetric group, permutation groups, maximal subgroups.

this paper, we examine an extension of the question answered in the computational group theory course: the case when one or both of the groups in Suschanskiĭ's question is allowed to be an alternating group.

## 2. Preliminaries

We begin by stating the following three questions:

- Q1: For what ordered pairs $(n, m)$ does $S_m$ have a maximal subgroup that is isomorphic to $A_n$?
- Q2: For what ordered pairs $(n, m)$ does $A_m$ have a maximal subgroup that is isomorphic to $S_n$?
- Q3: For what ordered pairs $(n, m)$ does $A_m$ have a maximal subgroup that is isomorphic to $A_n$?

We can answer the first question immediately with the following easy proposition.

PROPOSITION 2.1. *Suppose that a symmetric group $S_n$ has a subgroup $H$ that can be generated by a subset that only contains elements of odd order. Then $H$ is a subgroup of the alternating group $A_n$.* □

Since any group isomorphic to an alternating group can be generated by the images of 3-cycles, this proves that the only time $S_m$ has a maximal subgroup isomorphic to $A_n$ is if $n = m$.

To answer the remaining two questions, we simply need to look at maximal subgroups of the alternating group $A_m$. We will answer these questions by finding families of ordered pairs, and then showing that there can be no other ordered pairs outside of these families. There will be seven families that compose the answer to Q2, and these will be denoted $\mathcal{F}(S)_i$; the families that answer Q3 will be denoted $\mathcal{F}(A)_i$.

We reviewie a few basic facts about the maximal subgroups of symmetric and alternating groups. The following is well-known, and is not difficult to show.

PROPOSITION 2.2. *Let $m > 2$, $X_m$ be either $S_m$ or $A_m$, and $M$ be a maximal subgroup of $X_m$. Then one of the following holds:*

- (a) *$M$ acts intransitively on $\{1, \ldots, m\}$ and $M \cong (S_k \times S_{m-k}) \cap X_m$, where $k \neq \frac{m}{2}$.*
- (b) *$M$ acts transitively but imprimitively on $\{1, \ldots, m\}$, $M \cong (S_r \wr S_k) \cap X_m$, where $kr = m$ and $k, r > 1$.*
- (c) *$M$ acts primitively on $\{1, \ldots, m\}$.* □

The cases where the maximal subgroup does not act primitively are relatively easy and can be dealt with immediately. Suppose that $A_m$ has a maximal subgroup $M$ that is isomorphic to $S_n$, and that $M$ acts intransitively on $\{1, \ldots, m\}$. Then $M$ has the structure from Proposition 2.2(a), and it is an easy exercise to see that $M$ must lie in the following family:

$$\mathcal{F}(S)_1 := \{(n, n+2) \mid n \geq 3\}.$$

The only case where $A_m$ has a maximal subgroup that is isomorphic to a symmetric group $S_n$ that acts transitively but imprimitively on $\{1, \ldots, m\}$ is when $(n, m) = (4, 6)$. This ordered pair is already in $\mathcal{F}(S)_1$, although that instance represented an intransitive maximal subgroup isomorphic to $S_4$.

Now suppose that $A_m$ has a maximal subgroup $M$ that is isomorphic to $A_n$, and that $M$ does not act transitively on $\{1, \ldots, m\}$. Then $M$ has the structure from part (a) of Proposition 2.2, and it is again an easy exercise to see that $M$ must lie in the following family:

$$\mathcal{F}(A)_1 := \{(n, n+1) \mid n \geq 3\}.$$

Finally, note that an alternating group $A_n$ for $n \geq 5$ can never have the form of the wreath product from part (b) of Proposition 2.2, since the wreath product is not simple. For $n < 5$, we may check the cases individually to see that there are no maximal subgroups of the form described in part (b) that answer Q3.

## 3. The primitive case

For the remainder of the paper, we will be considering a subgroup $X_n$ of $A_m$ such that $X_n$ is isomorphic to $S_n$ or $A_n$, and that acts primitively on $\{1, \ldots, m\}$. Then $X_n$ is in family (f) from [**4**] (all of the families (a) through (f) are listed after the following paragraph), and is therefore maximal in $A_m$ unless one of following holds:

(1) $n = 6$ and $X_n < M \leq Aut(S_n)$, where $M$ also embeds into $A_m$.
(2) $X_n \cong A_n$, and $X_n$ is contained in the image of $S_n$ in $A_m$.
(3) The pair $(n, m)$ is explicitly listed as an exception in [**4**].

It remains to determine exactly when $S_n$ and $A_n$ act primitively on a set of cardinality $m \neq n$. To do this, we assume $X_n$ acts primitively and we look at a point stabilizer $H$ in $X_n$. Because the action of $X_n$ on $\{1, \ldots, m\}$ is primitive, $H$ is maximal in $X_n$, and $m = |X_n : H|$. The possibilities for $H$ were enumerated in [**4**]:

(a) $H \cong (S_k \times S_{n-k}) \cap X_n$ where $k \neq \frac{n}{2}$ (the intransitive case).
(b) $H \cong (S_r \wr S_k) \cap X_n$ where $n = kr$ and $k, r > 1$ (the imprimitive case).
(c) $H \cong AGL(k, p) \cap X_n$ where $n = p^k$ and $p$ prime (the affine case).
(d) $H \cong (T^k.(Out(T) \times S_k) \cap X_n$ where $T$ is a nonabelian simple group, $k \geq 2$, and $n = |T|^{k-1}$ (the diagonal case).
(e) $H \cong (S_r \wr S_k) \cap X_n$ where $n = r^k$, $r \geq 5$, and $k > 1$ (the wreath case).
(f) $T \lhd H \leq Aut(T)$ with $T$ a nonabelian simple group, $T \neq A_n$, and $H$ acts primitively on $\{1, \ldots, n\}$ (the almost simple case).

Moreover, [**4**] states that any subgroup of $X_n$ of one of these forms is maximal, save for a list of explicit exceptions. The action of $X_n$ on the cosets of a maximal subgroup yields a primitive action, and so we may simply consider the action of $X_n$ on subgroups of the six forms listed above. We now only need to determine the values of $n$ where $S_n$ (respectively $A_n$) has a maximal subgroup of each type, taking into account the exceptions listed in [**4**]. Once again, Magma was useful in working with these exceptions.

Note that for cases (c)–(f), $S_n$ embeds in $A_m$ rather than the more general $S_m$ by results from [**6**]. Therefore, if $H$ is of one of these four types with $H \not\leq A_n$, then the image of $A_n$ will always be contained in the image of $S_n$ in $A_m$. In this case, the image of $A_n$ will never be maximal. Similarly, we may conclude in cases (a) and (b) that $A_m$ has no maximal subgroup isomorphic to $A_n$ if the image of $S_n$ embeds into $A_n$.

We will examine the six cases for $H$ individually.

**3.1. The intransitive case.** Suppose now that $H$ is intransitive. If $X_n \cong S_n$, then $H \cong S_k \times S_{n-k}$ $(k \neq n/2)$, and $m = |S_n : H| = \binom{n}{k}$. The results from [**2, 3**] and the exceptions from [**4**] tell us that we get the following maximal embeddings of $S_n$ into $A_m$:

$$\mathcal{F}(S)_2 := \left\{ \left(n, \binom{n}{k}\right) \;\middle|\; \begin{array}{l} 2 \leq k \leq \frac{n}{2} - 1, \binom{n-2}{k-1} \text{ is even,} \\ \left(n, \binom{n}{k}\right) \neq (6,15),\ (10,120),\ (12,495) \end{array} \right\}.$$

If $X_n \cong A_n$, then $H \cong (S_k \times S_{n-k}) \cap A_n$ and $m = |A_n : H| = \binom{n}{k}$. This will not be maximal if the image of $S_n$ is contained in $A_m$, which leaves:

$$\mathcal{F}(A)_2 := \left\{ \left(n, \binom{n}{k}\right) \;\middle|\; 2 \leq k \leq \frac{n}{2} - 1, \binom{n-2}{k-1} \text{ is odd} \right\}.$$

**3.2. The imprimitive case.** Suppose now that $H$ is transitive but imprimitive. If $X_n \cong S_n$, then $H \cong S_r \wr S_k$ $(kr = n$, and $k, r > 1)$, and $m = |S_n : H| = \frac{(kr)!}{(r!)^k k!}$. The results from [**6**] and the exceptions from [**4**] tell us that we get the following maximal embeddings of $S_n$ into $A_m$:

$$\mathcal{F}(S)_3 := \left\{ \left(kr, \frac{(kr)!}{(r!)^k k!}\right) \;\middle|\; k, r > 1,\ k + r \geq 6,\ (k, r) \notin \mathcal{C} \right\},$$

where $\mathcal{C}$ was defined in the introduction.

If $X_n \cong A_n$, then $H \cong (S_r \wr S_k) \cap A_n$ $(kr = n$, and $k, r > 1)$, and

$$m = |A_n : H| = \frac{(kr)!/2}{(r!)^k k!/2} = \frac{(kr)!}{(r!)^k k!}.$$

This will not be maximal if the image of $S_n$ is contained in $A_m$, which leaves:

$$\mathcal{F}(A)_3 := \left\{ \left(kr, \frac{(kr)!}{(r!)^k k!}\right) \;\middle|\; k, r > 1,\ k + r \geq 6,\ (k, r) \in \mathcal{C} \right\}.$$

**3.3. The affine case.** Suppose now that $H \cong AGL(k, p)$ is affine; then $n$ is equal to $p^k$. We will use the fact that the affine general linear group $AGL(k, p)$ is contained $A_{p^k}$ iff $p = 2$.

If $X_n \cong S_{p^k}$, then $S_{p^k}$ only has a maximal affine subgroup if $p$ is odd. There is an infinite family of exceptions listed in [**4**] that occur when $k = 1$. This gives us an infinite family:

$$\mathcal{F}(S)_4 := \left\{ \left(p^k, |S_{p^k} : AGL(k, p)|\right) \;\middle|\; p \text{ is an odd prime, } k > 1 \right\}.$$

If $X_n \cong A_{p^k}$, then $X_n$ will always be contained in the image of $S_{p^k}$ unless $p = 2$. Then $X_n \cong A_{2^k}$ embeds maximally in $A_m$ in the following conditions:

$$\mathcal{F}(A)_4 := \left\{ \left(2^k, |A_{2^k} : AGL(k, 2)|\right) \;\middle|\; k \geq 3 \right\}.$$

**3.4. The diagonal case.** Suppose now that $H$ is diagonal, let $T$ be a non-abelian simple group, let $k \geq 2$, and let $D = T^k.(Out(T) \times S_k)$. Then $n = |T|^{k-1}$ and $H \cong D$. We would like to be able to determine exactly when this $H$ is contained in $A_n$, but we only have an incomplete answer. It is known that $H$ can lie outside of $A_n$ iff one of the following occurs:

(1) $k = 2$ and $|\{t \in T \mid t^2 = 1\}| \equiv 2 \pmod 4$.

(2) $k > 2$, and $Out(T)$ contains an automorphism of $T$ that acts of $T^{k-1}$ via an odd permutation.

If $X_n \cong S_{|T|^{k-1}}$ and $D$ contains an odd permutation of $T^{k-1}$, then $m = |S_n : D|$. This yields:

$$\mathcal{F}(S)_5 := \left\{ \left( |T|^{k-1}, |S_{|T|^{k-1}} : D| \right) \;\middle|\; k \geq 2,\ D \text{ contains an odd permutation} \right\}.$$

If $X_n \cong A_{|T|^{k-1}}$ and $D$ contains only even permutations of $T^{k-1}$, then $m$ is equal to $|A_n : D|$. This yields:

$$\mathcal{F}(A)_5 := \left\{ \left( |T|^{k-1}, |A_{|T|^{k-1}} : D| \right) \;\middle|\; k \geq 2,\ D \text{ contains only even permutations} \right\}.$$

There were no exceptions in [**4**] for this case.

**3.5. The wreath case.** Suppose now that $H \cong S_r \wr S_k$ for $n = r^k$, $r \geq 5$, $k > 1$. Then $H$ always contains an odd permutation, and is never a subgroup of $A_n$; then the image of $A_n$ will always be contained in the image of $S_n$ and will never be maximal. We conclude that $m = |S_n : H| = \frac{(r^k)!}{(r!)^k k!}$, and we get the following two families of ordered pairs:

$$\begin{aligned} \mathcal{F}(S)_6 &:= \left\{ \left( r^k, \frac{(r^k)!}{(r!)^k k!} \right) \;\middle|\; r \geq 5,\ k > 1 \right\} \\ \mathcal{F}(A)_6 &:= \emptyset. \end{aligned}$$

There were no exceptions in [**4**] for this case.

**3.6. The almost simple case.** Suppose now that $H$ is almost simple. We would like to determine when $H < A_n$, but this is currently an intractable problem. However, we can provide an implicit solution. Note that all simple groups are generated by their elements of odd order. Then by Proposition 2.1, all simple groups must be contained in alternating groups. Other almost simple groups, however, can lie outside of the alternating group.

We now consider the exceptions from [**4**] for the almost simple case. Define the four sets of exceptions as follows:

$$\begin{aligned} \mathcal{E}(S) &:= \left\{ (8,120), (10,2520), (22, |A_{24} : M_{24}|) \right\} \\ X(A) &:= \left\{ \begin{array}{l} (7,15),\ (9,120),\ (11,2520),\ (23, |A_{24} : M_{24}|), \\ (175, |A_{176} : HS|),\ (275, |A_{276} : \mathrm{Co}_3|) \end{array} \right\} \\ \mathcal{I}(A)_1 &:= \left\{ \left( c-1, |A_c : \mathrm{Sp}(2d,2)| \right) \;\middle|\; c = 2^{2d-1} \pm 2^{d+1},\ d \geq 3 \right\} \\ \mathcal{I}(A)_2 &:= \left\{ \left( 2^d - 1, |A_{2^d} : \mathrm{AGL}(d,2)| \right) \;\middle|\; d \geq 3 \right\}. \end{aligned}$$

For convenience, define $\mathcal{E}(A) = X(A) \cup \mathcal{I}(A)_1 \cup \mathcal{I}(A)_2$. Then we get two more families of ordered pairs:

$$\begin{aligned} \mathcal{F}(S)_7 &:= \left\{ \left( n, |S_n : H| \right) \;\middle|\; \begin{array}{l} H \text{ almost simple, primitive,} \\ H \not\leq A_n,\ \left( n, |S_n : H| \right) \notin \mathcal{E}(S) \end{array} \right\}, \\ \mathcal{F}(A)_7 &:= \left\{ \left( n, |A_n : H| \right) \;\middle|\; \begin{array}{l} H \text{ almost simple, primitive,} \\ H \leq A_n,\ \left( n, |A_n : H| \right) \notin \mathcal{E}(A) \end{array} \right\}. \end{aligned}$$

## 4. Conclusion

Save for the implicit definitions the diagonal and almost simple cases, we may now explicitly answer the original three questions posed in this paper. These are complete lists of solutions to Q2 and Q3 because we have exhausted every possible type of maximal subgroup of symmetric groups and alternating groups by [**4**].

THEOREM 4.1. *The set of all ordered pairs* $(n, m)$ *such that* $S_m$ *has a maximal subgroup that is isomorphic to* $A_n$ *is exactly*

$$\{(n, n) \mid n \geq 2\}.$$

THEOREM 4.2. *The set of all ordered pairs* $(n, m)$ *such that* $A_m$ *has a maximal subgroup that is isomorphic to* $S_n$ *is exactly*

$$\bigcup_{i=1}^{7} \mathcal{F}(S)_i.$$

THEOREM 4.3. *The set of all ordered pairs* $(n, m)$ *such that* $A_m$ *has a maximal subgroup that is isomorphic to* $A_n$ *is exactly*

$$\bigcup_{i=1}^{7} \mathcal{F}(A)_i.$$

## 5. Acknowledgements

This work was inspired by a computational group theory course taught by Professor Nigel Boston at the University of Wisconsin-Madison, and the author would like to thank him for his guidance and encouragement. Additionally, the author would like to thank Jack Schmidt for generalizing the author's original version of Proposition 2.1; this generalization was helpful in the almost simple case. Finally, the author would like to thank Benjamin Newton, who was extremely helpful in getting these results.

## References

[1] W. Bosma, J. Cannon, and C. Playoust. *The* MAGMA *algebra system. I. The user language.* J. Symbolic Comput., **24**(3-4):235–265, 1997.
[2] E. Halberstadt, *On Certain Maximal Subgroups of Symmetric or Alternating Groups*, Math Z. **151** (1976), 117–125.
[3] L. A. Kalužnin and M. H. Klin, *Certain Maximal Subgroups of Symmetric and Alternating Groups*, Math. Sb. **87** (1972), 91–121.
[4] M. W. Liebeck, C. E. Praeger, J. Saxl, *A Classification of the Maximal Subgroups of the Finite Alternating and Symmetric Groups*, J. Algebra **111** (1987), 365–383.
[5] V. D. Mazurov and E. I. Khukhro, *Unsolved Problems in Group Theory: The Kourovka Notebook*, 16th Ed., Novosibirsk, 2006.
[6] B. Newton and B. Benesh, *A Classification of Certain Maximal Subgroups of Symmetric Groups*, J. Algebra **304** (2006), 1108–1113.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, ONE OXFORD STREET, CAMBRIDGE, MASSACHUSETTS 02138

*E-mail address*: benesh@math.harvard.edu