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A classification of certain maximal subgroups of symmetric groups

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Abstract

Problem 12.82 of the Kourovka Notebook asks for all ordered pairs (n, m) such that the symmetric group S_n embeds in S_m as a maximal subgroup. One family of such pairs is obtained when $m = n + 1$. Kalužnin and Klin [L.A. Kalužnin, M.H. Klin, Certain maximal subgroups of symmetric and alternating groups, Math. Sb. 87 (1972) 91–121] and Halberstadt [E. Halberstadt, On certain maximal subgroups of symmetric or alternating groups, Math. Z. 151 (1976) 117–125] provided an additional infinite family. This paper answers the Kourovka question by producing a third infinite family of ordered pairs and showing that no other pairs exist.

1. Introduction

In problem 12.82 of the Kourovka Notebook [5], V.I. Suschanskiĭ poses the question of describing the set $\mathcal{M}$ of all pairs of positive integers (n, m) such that the symmetric group S_m contains a maximal subgroup isomorphic to S_n.

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One obvious family of such pairs is

$$\{(n, n+1) \mid n \geqslant 1\},$$

which we will refer to as 'Family 1.'

A second family can be constructed by considering the action of S_n on the set of subsets of $\{1, 2, \dots, n\}$ of size k. If $1 \leqslant k < n$, this gives an embedding of S_n into $S_{\binom{n}{k}}$. This method was described by Kalužnin and Klin [3], along with certain conditions under which the embedding is maximal. This work, together with refinements made by Halberstadt [2], gives the following:

$$\left\{(n, m) \;\middle|\; m = \binom{n}{k},\ 2 \leqslant k \leqslant \frac{n}{2} - 1,\ \text{and } \binom{n-2}{k-1} \text{ is odd}\right\} \subseteq \mathcal{M}.$$

We shall refer to this subset of $\mathcal{M}$ as 'Family 2.'

This paper gives a third infinite family of pairs in $\mathcal{M}$, and shows that the three families described yield all possible maximal embeddings of one symmetric group into a larger one.

2. Preliminaries

We begin by reviewing a few basic facts about the maximal subgroups of symmetric groups. The following is well known, and is not difficult to show.

Proposition 2.1. *Let $n > 2$, and let H be a maximal subgroup of the symmetric group S_n.*

(a) *If H is intransitive, then $H \cong S_a \times S_b$, where a and b are positive integers such that $a + b = n$ and $a \neq b$.*
(b) *If H is transitive but imprimitive, then $H \cong S_r \wr S_k$, where $kr = n$ and $k, r > 1$.*

It is also fairly easy to demonstrate that every subgroup of S_n matching one of the descriptions in Proposition 2.1 is maximal.

Note that none of the subgroups of S_n described by Proposition 2.1 can be isomorphic to a symmetric group except for intransitive subgroups of the form $S_{n-1} \times S_1$. All such subgroups appear in Family 1. It follows that any pairs in $\mathcal{M}$ not listed in Family 1 must involve primitive embeddings of S_n into S_m.

So let us suppose that S_n acts primitively and faithfully on some set Ω. Then the stabilizer H of a point in Ω is maximal in S_n. Proposition 2.1 may thus be used to analyze the forms that H may take.

3. Families in $\mathcal{M}$

We first consider the case where the maximal point stabilizer H is intransitive. We have that $H \cong S_k \times S_{n-k}$, with $k \neq n/2$. As such, H is precisely the stabilizer of a point in the action of S_n on k-element subsets of $\{1, 2, \dots, n\}$. The actions considered in this case will thus yield exactly the same set of pairs in $\mathcal{M}$ as those listed in Family 2.

We now turn to the situation where H is transitive but imprimitive. In this case, H is isomorphic to $S_r \wr S_k$, with $k, r > 1$ and $kr = n$. The multiplicative action of S_n on the right cosets of H is faithful exactly when $\mathrm{core}_{S_n}(H) = 1$. This fails to be true when $(k, r) = (2, 2)$, but for all other $k, r > 1$, the action gives us a primitive embedding of S_n into $S_{f(k,r)}$, where we define

$$f(k, r) = |S_n : S_r \wr S_k| = \frac{(kr)!}{(r!)^k k!}. \tag{1}$$

In such situations, we let $G_{(k,r)}$ be the image of S_n under this embedding.

Excepting $G_{(2,3)}$ and $G_{(3,2)}$, the group $G_{(k,r)}$, where defined, is always a maximal subgroup of either $S_{f(k,r)}$ or $A_{f(k,r)}$. This is shown in the following theorem.

Theorem 3.1. *Let k and r be integers such that $k, r > 1$, and $k + r \geqslant 6$. If $G_{(k,r)} \leqslant A_{f(k,r)}$, then $G_{(k,r)}$ is maximal in $A_{f(k,r)}$. If $G_{(k,r)} \nleqslant A_{f(k,r)}$, then $G_{(k,r)}$ is maximal in $S_{f(k,r)}$.*

Proof. We have that $G_{(k,r)}$ is a primitive subgroup of $S_{f(k,r)}$. Furthermore, since $k, r \geqslant 2$ and $k + r \geqslant 6$, it follows that $kr \geqslant 8$ and that $G_{(k,r)} \cong S_{kr}$ is almost simple. Thus $G_{(k,r)}$ is a subgroup of $S_{f(k,r)}$ of type (f), as described in [4]. According to the main theorem of that paper, if K is a subgroup of S_d of type (f) and L is a subgroup of S_d such that $K < L < KA_d$, then L is almost simple and either $\mathrm{Soc}(K) = \mathrm{Soc}(L)$, or K is in an explicit list of exceptional cases.

But $G_{(k,r)} \cong S_{kr}$ where $kr > 6$, so $\mathrm{Soc}(G_{(k,r)}) \cong A_{kr}$ and $G_{(k,r)} \cong \mathrm{Aut}(A_{kr})$. Thus no almost simple group L exists such that $G_{(k,r)} < L$ and $\mathrm{Soc}(L) = \mathrm{Soc}(G_{(k,r)})$.

It therefore remains only to verify that $G_{(k,r)}$ does not appear in the list of exceptional cases in [4]. We need only consider groups of type (f) on this list which possess a socle isomorphic to an alternating group of degree at least 8. The paper lists four infinite families of such groups. Twelve additional groups of this description are also given. In the cases where the socle of such a group is isomorphic to A_{kr} for integers k and r satisfying the hypotheses of the theorem, we may compare the degree of the non-maximal subgroup listed to $f(k, r)$, which is the degree of the symmetric group into which $G_{(k,r)}$ embeds. In none of these cases do the two degrees coincide. We therefore have that $G_{(k,r)}$ is maximal in $G_{(k,r)}A_{f(k,r)}$, and the result follows. $\square$

We now wish to determine the circumstances under which $G_{(k,r)}$ is contained in $A_{f(k,r)}$. Succinct conditions for this occurrence can be found using a method similar to the one used in [3, §6, Lemma 2].

For this purpose it will be useful to describe in a concrete manner the set Ω on which S_n acts. For $k, r > 1$, with $kr = n$, let Ω be the set of partitions of $\{1, 2, \dots, n\}$ into k disjoint subsets, each of cardinality r. If $(k, r) \neq (2, 2)$, then S_n acts primitively and faithfully on this set. Note that $|\Omega| = f(k, r)$, and the stabilizer in S_n of a point in Ω has exactly the form under consideration, namely $S_r \wr S_k$.

Consider an arbitrary transposition $(\alpha\ \beta)$ in S_{kr}. The permutation of Ω induced by $(\alpha\ \beta)$ may be expressed as the product of disjoint transpositions in $S_{f(k,r)}$. Note that the number of transpositions in this product is independent of our choice of α and β, and is a function of k and r alone. Let $c(k,r)$ be this function.

We may compute $c(k,r)$ as one half of the number of elements of Ω which are not fixed by the action of $(\alpha\ \beta)$. So $c(k,r)$ is equal to the number of partitions of $\{1,\dots,n\}$ in which α and β appear in distinct parts. We have that

$$c(k,r)=\frac{1}{2}\binom{kr-2}{r-1}\binom{kr-r-1}{r-1}f(k-2,r)=\frac{(kr-2)!r^2k(k-1)}{2(r!)^k k!}. \tag{2}$$

(For convenience, we define $f(0,r)$ to be 1.)

As S_{kr} is generated by its transpositions, $G_{(k,r)}$ is contained in $A_{f(k,r)}$ if and only $c(k,r)$ is even. An analysis of Eq. (2) yields the following.

Proposition 3.2. *Let k and r be integers such that $k,r>1$. Then $c(k,r)$ is odd if and only if k and r satisfy one of the following conditions*:

(i) $k=2$, *and* $r=2^d+1$ *for some integer* $d\geqslant 0$.
(ii) $k=3$ *and* $r=2^d+1$ *for some integer* $d\geqslant 1$.
(iii) $k\geqslant 4$ *is even, and* $r=2$.

Finally, combining this result with Theorem 3.1, we obtain a description of a family of ordered pairs in $\mathcal{M}$.

Corollary 3.3. *Let k and r be integers such that $k,r>1$ and $k+r\geqslant 6$. Suppose that k and r also satisfy one of the conditions listed in Proposition* 3.2. *Then* $(kr, f(k,r))\in\mathcal{M}$.

The set of pairs obtained from Corollary 3.3 will be referred to as 'Family 3.'

4. The primitive case

In this section, we will prove that there are no ordered pairs in $\mathcal{M}$ outside of the three families already described. We have already considered cases where the maximal point stabilizer H is either intransitive or transitive but imprimitive. It remains to check the case where H is primitive. The following lemma goes most of the way toward that end. Its proof was supplied to the authors by Jan Saxl.

Lemma 4.1. *Let $n>8$, and suppose that H is a primitive subgroup of S_n that does not contain A_n. Then the multiplicative action of S_n on the set of right cosets of H in S_n involves only even permutations of this set.*

Proof. Let $m=|S_n:H|$. Let $x\in S_n$ be a transposition, and consider the permutation of the m right cosets of H in S_n induced by x. This permutation is of order 2, so it may be represented as the product of disjoint transpositions in S_m. Let l be the number of transpositions in this product. Since S_n is generated by its transpositions, it suffices to show that l is even. We assume toward a contradiction that l is odd.

By a theorem of Jordan (see [1, Theorem 3.3A]), the subgroup H contains no transpositions, so for every $g \in S_n$, we have that $x^g \notin H$. Thus the permutation induced by x is without fixed points. It follows that $m = 2l$.

Now take $P \in \mathrm{Syl}_2(H)$ and $T \in \mathrm{Syl}_2(S_n)$ such that $P \leqslant T$. Note that since l is odd, $|T : P| = 2$. Let $y \in T$ be a 4-cycle. Then $y^2 \in P \subseteq H$ is a permutation that fixes all but 4 points. By further results of Jordan (see [1, Theorem 3.3A, Example 3.3.1]), however, H contains no such element. With this contradiction, the proof is complete. □

The main result now follows easily.

Theorem 4.2. *The set $\mathcal{M}$ is the union of Family* 1*, Family* 2*, and Family* 3.

Proof. As noted above, it remains only to consider the situation where S_n is acting faithfully on a set in such a way that the point stabilizer H of the action is maximal in S_n and primitive. We must show that no such action yields a maximal embedding of S_n into $S_{|S_n:H|}$ that is not already accounted for in Families 1, 2 and 3.

No maximal embeddings arise when $A_n \leqslant H$, so we may assume that this is not the case. Thus when $n > 8$, we have by Lemma 4.1 that S_n embeds via this action into $A_{|S_n:H|}$, and does not embed maximally in $S_{|S_n:H|}$.

For $n \leqslant 8$, we may examine all primitive maximal subgroups H of S_n individually. In only one case does the action associated with H yield a maximal embedding, and this is the primitive embedding of S_5 into S_6. But the pair $(5, 6)$ is already contained in Family 1, and thus the proof is complete. □

As a final thought, it is interesting to note that when the action of S_n on the cosets of a primitive maximal subgroup H gives an embedding of S_n into A_m (where $m = |S_n : H|$), the image of S_n is almost always maximal in A_m. This can be seen by appealing to [4], as in the proof of Theorem 3.1. The only exceptions occur when $(n, m) \in \{(7, 120), (8, 120), (10, 2520), (22, |A_{24} : M_{24}|)\}$.

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