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# Games on Dihedral Groups

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# Games on Dihedral Groups


## Emily Twardy

**Advised by Dr. Bret Benesh**


April, 2021

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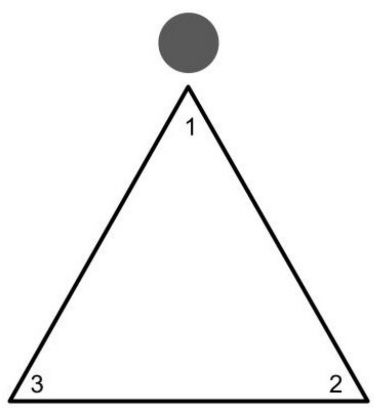
# Chapter 1

# Introduction

This thesis deals with two games (Return and Move) for two players: Alpha, who always goes first, and Beta, who always goes second. Let us get an intuition with an example before we formally define the game. We start by selecting a regular polygon, in this case, a triangle. Next, we label each vertex with a number such that Vertex 1 is at the top, and the numbers continue sequentially clockwise. Then we place a token on Vertex 1.

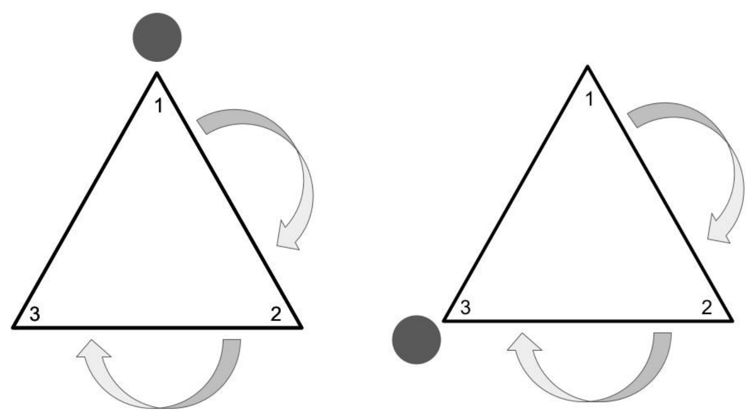



Alpha and Beta take turns using the symmetries of the triangle to move the token to different Vertices. For the triangle, the symmetries (the possible moves) are three rotations - rotation by $0°$ (the identity), $120°$, and $240°$ clockwise - and three flips, over Vertices 1, 2, and 3. Once we have the triangle and the symmetries of the triangle (i.e. the group), the game follows as such:

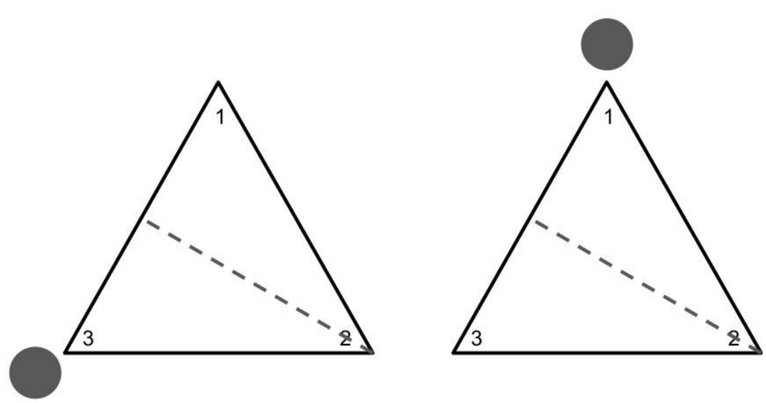
1. We put a token on Vertex 1.

2. Alpha chooses a symmetry, which may move to the token to different vertex.
3. Beta chooses a different symmetry, and the token might move again.
4. Play repeats like this, with each player choosing a symmetry that has not already been selected, until there are not more symmetries left.

For our first version of the game, Return, Alpha, wins only if the token ends on Vertex 1. Beta, wins only if the token ends up somewhere else. For our second game, move, Beta wins only if the token ends the game back at Vertex 1 and Alpha wins only if the token ends up somewhere else. As an example of one round, Alpha might start by rotating token around the triangle 240° clockwise, putting the token on Vertex 3.




Beta might then flip the token over Vertex 2, putting the token back on Vertex 1.



The players continue until all symmetries have been used exactly once. In this thesis, we seek to find out which player has a winning strategy for each version of the game for all regular polygons. We look at some examples of the game in Chapter 3.

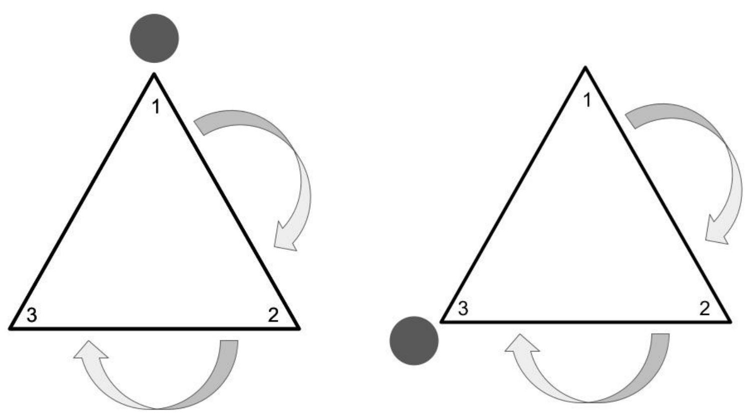
# Chapter 2

# Preliminaries

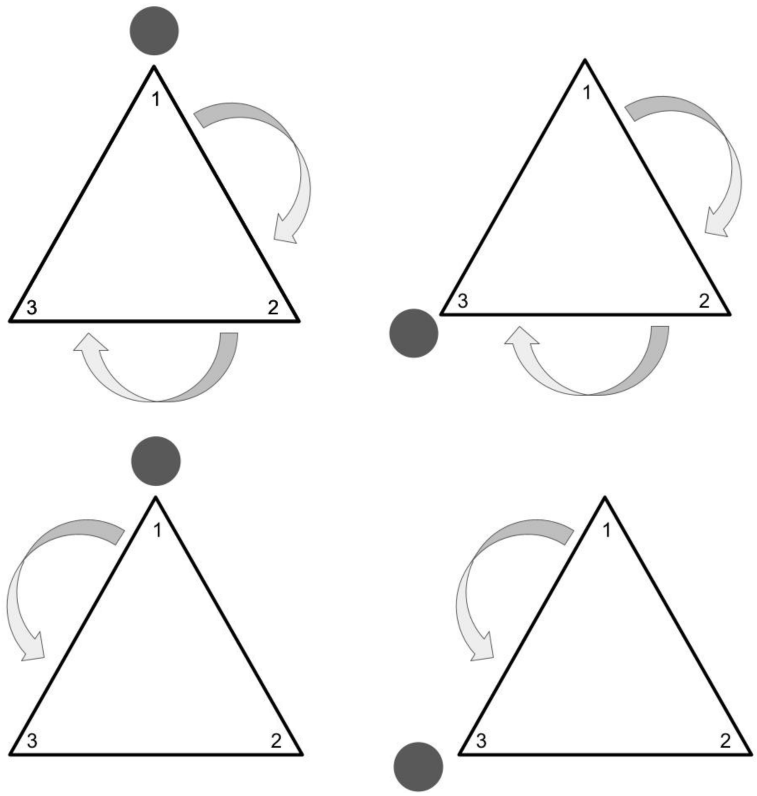
Before we define the game more formally, we need to have some definitions. The symmetries of regular polygons are governed by algebraic structures of groups.

**Definition 1** (group)**.** *A nonempty set $G$ is a group under an operation iff $G$ is closed under that operation and there exists $e \in G$ such that for all $x \in G$, $xe = ex = x$ and there exists $x' \in G$ such that $xx' = x'x = e$ and for all $x, y, z \in G$, $(xy)z = x(yz)$.*

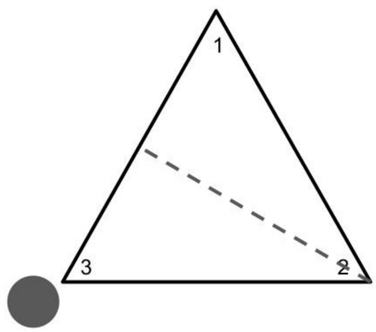
In particular, we will be looking at dihedral groups. Most simply, a dihedral group, $D_n$ is the set of symmetries of a regular polygon with $n$ sides. $D_n$ has $n$ rotations and $n$ mirror reflections, or flips. A rotation, $r$ is the smallest rotation of the polygon clockwise, such that the marker at Vertex 1 is now at Vertex 2. A rotation $r^a$ is that size of rotation $a$ times, sending sending the marker at Vertex $b$ to Vertex $a + b$. See this example of the rotation $r^2$.

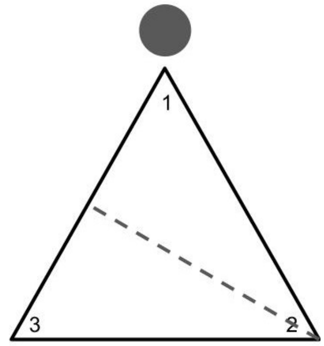


Note that the counterclockwise rotation $r^{-1}$ for a triangle is the same as the rotation $r^2$.




More generally, we will consider the clockwise rotation $r^j$ to be the same as the counterclockwise rotation $r^{n-j}$ and all other equivalent rotations, such as $r^{j+kn}$ for all $k$. For flips, we can call the flip which fixes a Vertex $j$ as the flip $f_j$. For even $n$, the flip $f_j$ can also be called the flip $f_(j + n/2)$, since Vertex $j + n/2$ would also be fixed by $f_j$. Likewise, the flip $f_j.5$ can also be called the flip $f_(j + n/2).5$.




We will briefly use cycle notation to define the effect an element of a dihedral group has on the vertices of a polygon. For example, $(1, 2, 3, 4)$ describes an element of a group that moves

- Vertex 1 to Vertex 2,
- Vertex 2 to Vertex 3,
- Vertex 3 to Vertex 4,
- and Vertex 4 to Vertex 1.

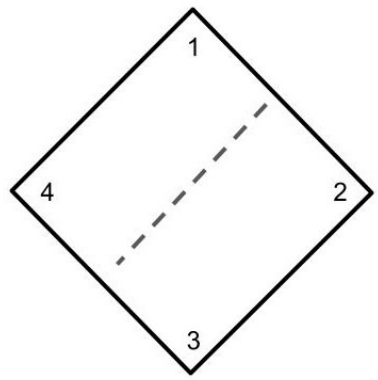
This would describe a 90 degree rotation of the square. Similarly, an element $(2, 5)(3, 4)$ would describe an element that fixes 1 (since it is not listed), swaps Vertices 2 and 5, and swaps Vertices 3 and 4; this describes $f_1$ for the regular pentagon.

Let $G$ be a dihedral group of order $2n$, denoted $D_n$. Then $G = \langle r, f\rangle$ where $r = (1, 2, \ldots, n)$ and $f = \prod_{i=1}^{\lfloor n/2\rfloor}(1-i, 1+i)$. Recall that the product $rf = fr^{-1}$.

We will use the following notation for the flips in $D_n$.

- $f_j := \prod_{i=1}^{\lfloor n/2\rfloor}(j-i, j+i)$, a flip that fixes $j$, and
- $f_{j.5} := \prod_{i=0}^{n/2-1}(j-i, j+i+1)$, a flip that fixes no element of $\{1, \ldots, n\}$, maps $j$ to $j+1$, and only exists if $n$ is even.

An example of the flip $f_{1.5}$.




We will be using exponentiation notation to describe what each group element does to the token. For instance, we would say the following for example above:

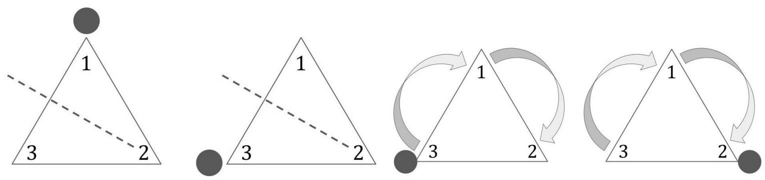
- $1^{f_{1.5}} = 2$
- $2^{f_{1.5}} = 1$
- $3^{f_{1.5}} = 4$
- $4^{f_{1.5}} = 3$

Note that $j^{f_j} = j$ for all integers $1 \leq j \leq n$ for all $D_n$, since $f_j$ is defined to fix $j$.

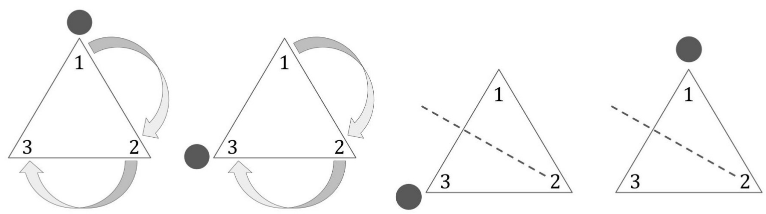
We can now define the game.

Let $G$ be a dihedral group. We define two games on $G$ for two players. We defined the players as Alpha and Beta, where Alpha is the first player to move in the game. In both games, the game starts with $X_0 := \emptyset$ and $g_0 := e$, the identity of $G$. On the $i+1$st move, the player picks $h_{i+1}$ from $G \setminus X_i$, defines $X_{i+1} := X_i \cup \{h_{i+1}\}$, and defines $g_{i+1} := g_i \cdot h_{i+1}$. The first game is called Return, and Alpha wins exactly when $1^{g_{|G|}} = 1$, and Beta wins only when $1^{g_{|G|}} \neq 1$, that is, the product of all the elements in order returns Vertex 1 to itself. The second game is called Move, and Alpha wins exactly when $1^{g_{|G|}} \neq 1$, and Beta wins only when $1^{g_{|G|}} = 1$. More simply, each round of the game looks at one possible sequence of all the elements of $G$ multiplied together.

Note that the game results in a product of elements, and the two players are working to determine the sequence of the elements, with the winner determined by whether the resultant move fixes 1. If the group is abelian, then the players choices don't matter. So the non-abelian-ness makes it interesting, and it means that the players' choices matter. Take for example, $f_2$ and $r^2$ in $D_3$. Does the order in which we play these two elements matter? The resultant move of $f_2 r^2$ is $f_3$, and the token lands on 2.




However, the resultant move of $r^2 f_2$ is $f_1$, and the token lands on 1.




So the order in which the elements are played matters.

## 2.1 Helpful lemmas

Later, we'll need to know what resultant move we get when we pair flips.

**Lemma 2.** *Let $G = D_{2n}$. Then*

1. $f_j f_{j+1} = r^2$ *if $n$ is odd*
2. $f_j f_{j+2} = r^4$ *if $n$ is odd*
3. $f_\ell f_{\ell+n/4} = r^{n/2}$ *if $n$ is even*

*for all $j \in \{1, 2, \dots, n\}$ and for all $\ell \in \{m/2 \mid m \in \mathbb{Z}, 2 \le m \le 2n+1\}$*

*Proof.* First, note that the product of two flips is a rotation, since each flip can be written as $fr^{j'}$, $f_j f_k = fr^{j'} fr^{k'} = ffr^{-j'} r^{k'} = r^{-j'+k'}$.

To determine which rotation, consider what happens to $j$ under $f_j f_{j+1}$:

$$j^{f_j f_{j+1}} = j^{f_{j+1}} = j + 2$$

since $f_j$ fixes $j$ and $f_{j+1}$ permutes $j$ and $j+2$.

Similarly:

$$j^{f_j f_{j+2}} = j^{f_{j+2}} = j + 4$$

since $f_j$ fixes $j$ and $f_{j+2}$ permutes $j$ and $j+4$. Thus, $f_j f_{j+2} = r^4$.

Now let $k = \ell + n/4$ and assume that $n$ is even. Note that $\ell$ is either an integer, or an integer plus 0.5. First, let's consider the case when $\ell$ is an integer. Then $\ell^{f_\ell f_k} = \ell^{f_k}$. Depending on whether $k$ is an integer or an integer plus 0.5, $f_k = \prod_{i=1}^{n/2}(\ell + n/4 - i, \ell + n/4 + i)$ or $f_k = \prod_{i=0}^{n/2-1}(\ell + n/4 - i, \ell + n/4 + i)$. The relevant transposition is when $i = n/4$: $(\ell + n/4 - n/4, \ell + n/4 + n/4) = (\ell, \ell + n/2)$. Then $\ell^{f_k} = \ell^{(\ell, \ell+n/2)} = \ell + n/2$. Thus, $f_\ell f_k = r^{n/2}$.

For when $\ell$ is not an integer, $\ell - 0.5$ is an integer. Then $(\ell - 0.5)^{f_\ell f_k} = \ell.5^{f_k}$. Likewise, depending on whether $k$ is an integer or an integer plus 0.5, $f_k = \prod_{i=1}^{n/2}(\ell.5 + n/4 - i, \ell.5 + n/4 + i)$ or $f_k = \prod_{i=0}^{n/2-1}(\ell.5 + n/4 - i, \ell.5 + n/4 + i)$. The relevant transposition is when $i = n/4$: $(\ell.5 + n/4 - n/4, \ell.5 + n/4 + n/4) = (\ell.5, \ell.5 + n/2)$. Then $\ell.5^{f_k} = \ell.5^{(\ell.5, \ell.5+n/2)} = \ell.5 + n/2$. Thus, $f_\ell f_k = r^{n/2}$.

□

Next, we can examine what happens when we switch the order of the flips.

**Lemma 3.** *Let $G = D_{2n}$ and $f_x, f_y \in G$ be flips. If $f_x f_y = r^j$ for some $j$, then $f_y f_x = r^{-j}$.*

*Proof.* Let $G = D_{2n}$ and $f_x, f_y \in G$ be flips. Since the product of two flips is a rotation, there exists $j$ such that $0 \leq j < n$ and $f_x f_y = r^j$. Note that $(f_x f_y)(f_y f_x) = f_x(f_y f_y)f_x = f_x f_x = e$, so $f_x f_y$ and $f_y f_x$ are inverses. Thus, if $f_x f_y = r^j$, then $f_y f_x = r^{-j}$, since $r^j$ and $r^{-j}$ are inverses. □

Next, we want to show the relationship between a rotation and fixing Vertex 1.

**Lemma 4.** *For $D_n$, If $g_{|G|} = r^a$, then $g_{|G|}$ fixes 1 iff $a \equiv 0 \mod n$.*

*Proof.* Let $g_{|G|} = r^a$. Then there exists $r^b \in D_n$ for $0 \leq b < n$ such that $r^a$ is the same as $r^b$. That is, $a \equiv b \mod n$. Since $r^b$ only fixes 1 when $r^b = e$, that is, when $b = 0$, then $r^{\pm a}$ fixes 1 only when $a \equiv 0 \mod n$. Thus, $g_{|G|}$ fixes 1 only when $a \equiv 0 \mod n$.

□

**Definition 5.** *We will sometimes want to look at a strategy that either Alpha or Beta could play. In this case, we will define the player implementing the strategy as Player and the other player as Opponent. In all cases,* $\{$*Player,Opponent*$\}$ *will equal* $\{$*Alpha,Beta*$\}$*. That is, we define Player and Opponent such that if Alpha is Player, then Beta is Opponent, and if Alpha is Opponent, then Beta is Player.*

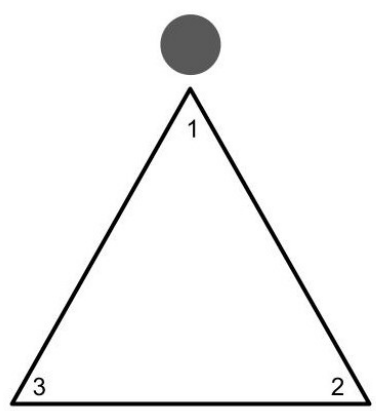
Before we generalize our results to all dihedral groups, we will look at some examples in the next chapter.

# Chapter 3

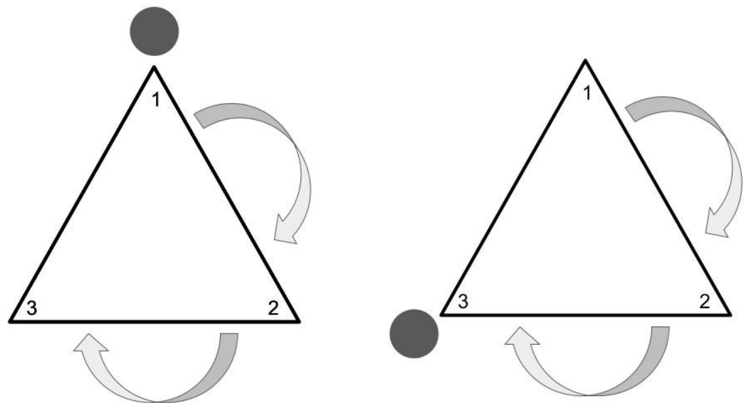
# Examples of Games

In this chapter, we will examine several concrete examples of how games could go. The purpose of these examples is to help the reader develop intuition about the game, as well as to demonstrate the ideas that we will prove later in a more concrete environment. We will look at examples involving the triangle, square, regular pentagon, and regular hexagon.

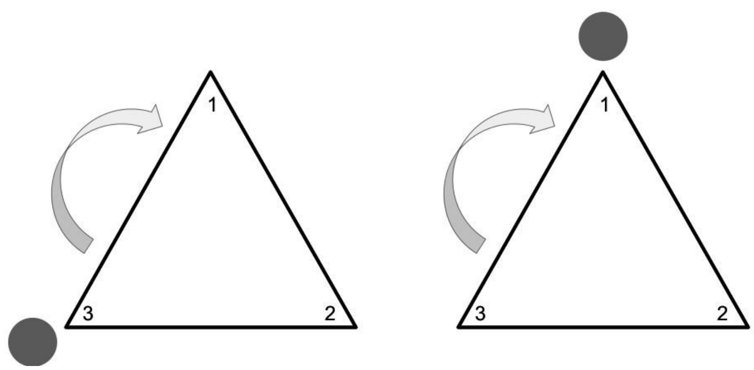
## 3.1 Example of a Triangle



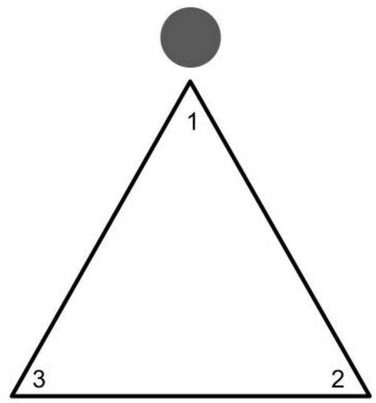

For the triangle, there are 6 moves to make: 3 rotations and 3 flips. We've defined these as $e$, the identity, or the rotation by zero degrees, $r^1$, the clockwise rotation by 120°, and $r^2$, the clockwise rotation by 240°. The flips are $f_1$, which switches 2 and 3 while fixing 1, $f_2$, fixing 2, and $f_3$, which fixes 3. Let's examine one possible outcome for the Return version of the game, where Alpha wants the marker to end up back at 1. Alpha might rotate the token around the triangle by playing $r^2$, putting the token on Vertex 3.



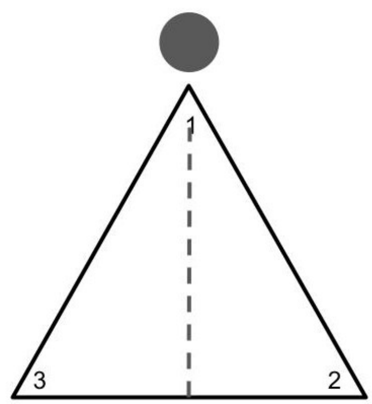

After that, Beta rotates the token around the triangle, playing $r^1$ putting the token back on Vertex 1. Note that the two rotations played are inverses of each other, so the product of those two rotations is $e$.



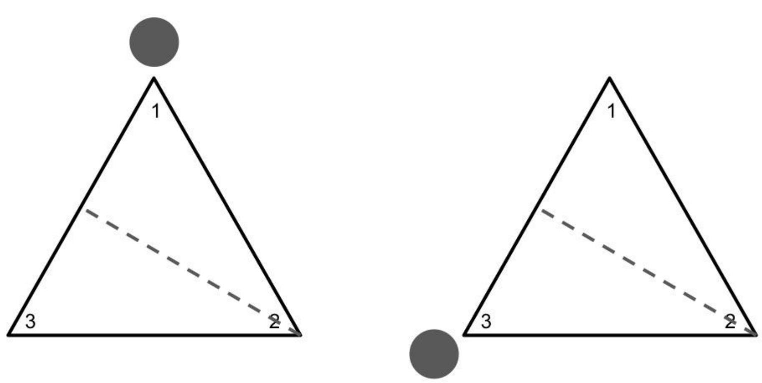

Alpha's goal is to end the token on 1, so they select $e$, leaving the token on 1.



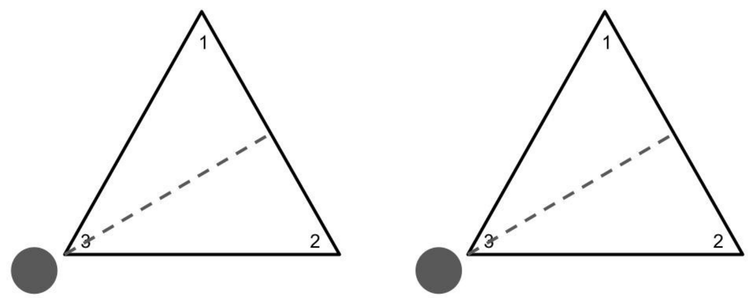

After that, Beta selects $f_1$, which still leaves the token on Vertex 1. The product of the last two moves is $f_1$.




Alpha might then flip the token over Vertex 2, putting the token on Vertex 3.



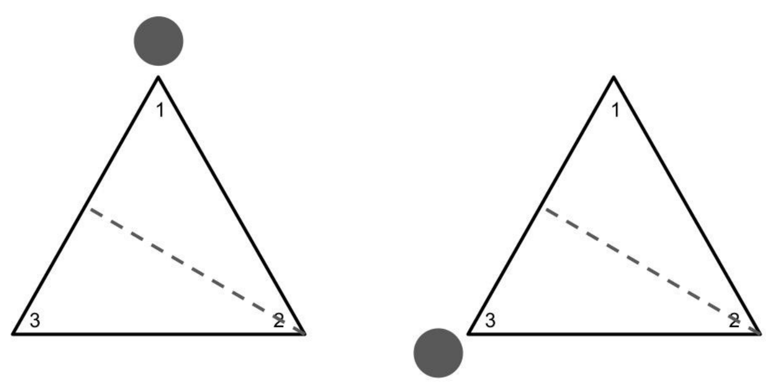
Beta only has one move left, the flip over 3, leaving the token on Vertex 3.



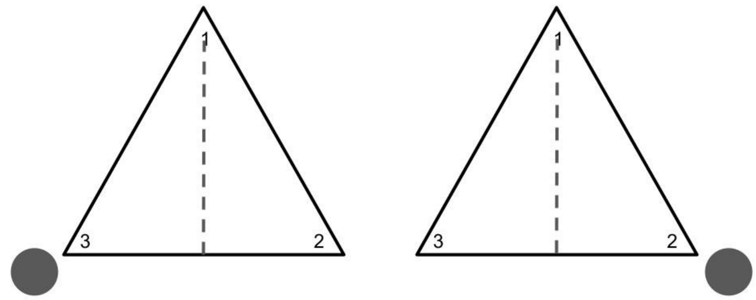

Beta wins this time. Notice that Beta paired $f_2$ with $f_3$ to make $r^2$. The product of all the moves in this order resulted in $f_1 r^2$, which did not fix 1. We will generalize this strategy in Chapter 4 to show that this strategy can work in $D_n$ for all $n \equiv 3 \mod 4$.

## 3.2 Triangle, Move Variation

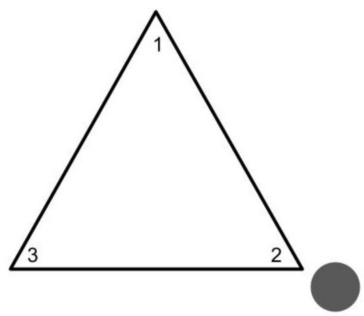
For Move, where Alpha wants the token to land not on Vertex 1, suppose Alpha wants to try a different strategy than Beta used. We'll see later why this is a bad idea. Alpha starts with $f_2$, which which puts the token on Vertex 3.



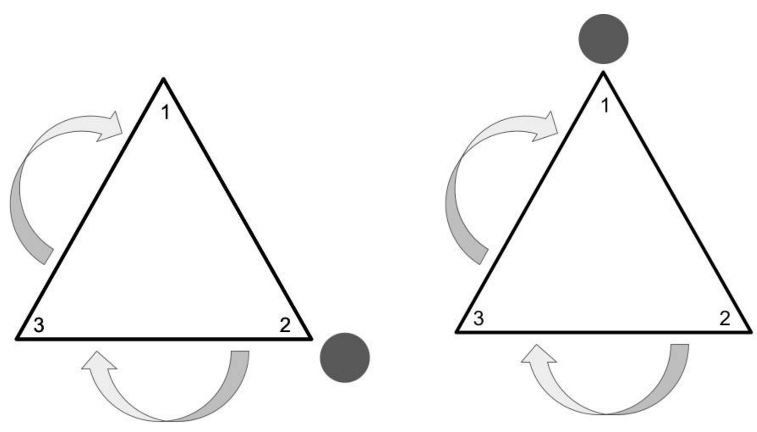

Beta chooses $f_1$, which puts the token on Vertex 2. The product of these two flips is $r^1$.



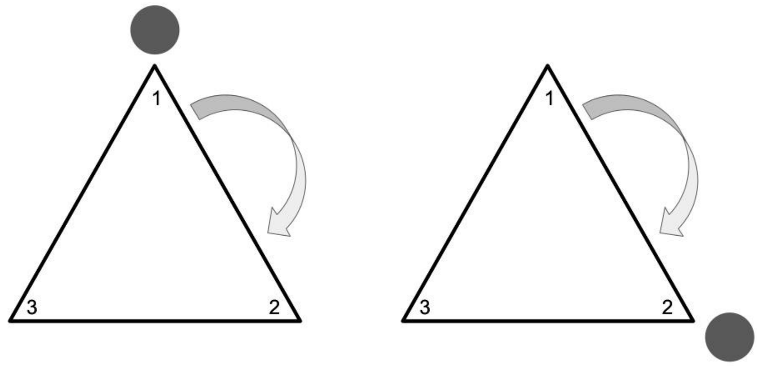

Alpha, seeing the token not on Vertex 1, chooses $e$. We'll see later that this was a bad move.



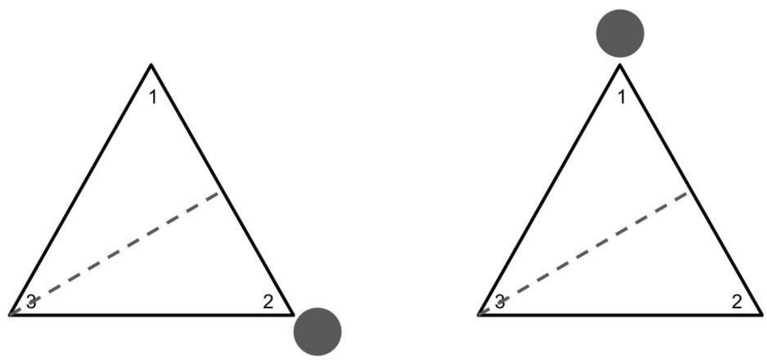

Beta might then rotate the token around the triangle, playing $r^2$, putting the token on Vertex 1.




After that, Alpha rotates the token around the triangle, playing $r^1$, putting the token back on Vertex 2. Notice that since $r^2$ and $r^1$ are inverses, Alpha's move "undoes" Beta's move. So the product of all the moves is $r^1 * e * e = r^1$

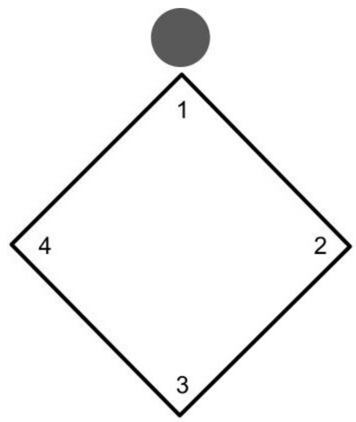



Beta only has one move left, the flip over 3, putting the token on Vertex 1.

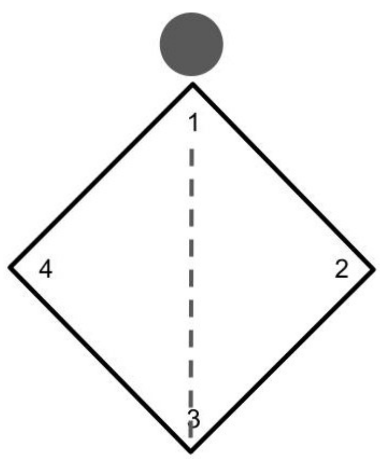



The product of all the moves in this order is $r^1 * f_3 = f_1$, which does not fix 1. Since the token landed on 1, Beta wins Move. We get a simple, yet important, insight from this example. When we had a different sequence of moves from the previous example and the token ended on Vertex 1, rather than Vertex 3, illustrating that player choice matters. We'll see later that Alpha could have won this game if they used Beta's strategy of pairing up the rotations and flips to get a final product of $f_1 r^{\pm 2}$.

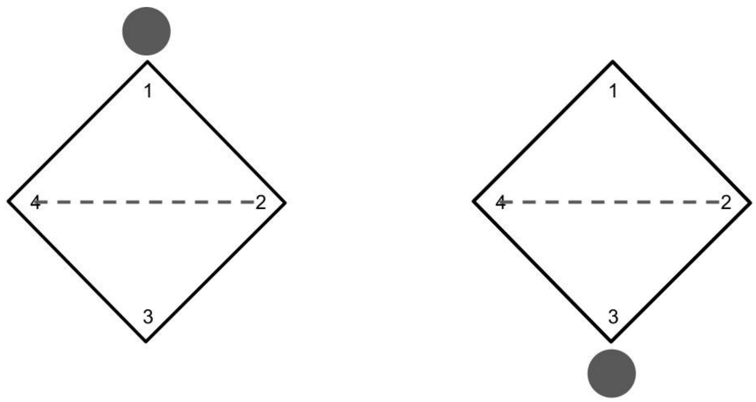
## 3.3 Example of a square



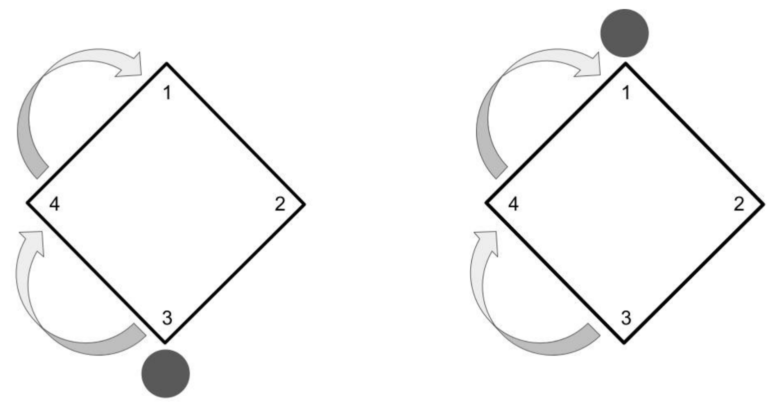

The possible moves for $D_4$ are the four rotations—$e$, $r^1$,$r^2$,$r^3$—and the four flips: $f_1$, $f_{1.5}$, $f_2$, $f_{2.5}$. For Return, Beta is going to try a strategy where they pair up the rotations with other rotations and the flips with other flips. Beta decides to pair up flips such that for any flip Alpha plays, Beta will pick the flip that results in the product of those two flips being $r^{n/2}$ which is the 180° rotation. If Alpha plays a rotation, Beta will play that rotation's inverse, to make the product $e$. Alpha starts with $f_1$.



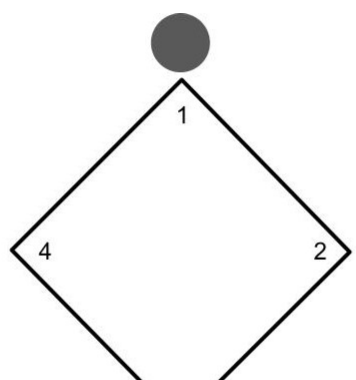

Beta chooses $f_2$, putting the token on Vertex 3. The product of $f_2$ and $f_3$ is $r^2$



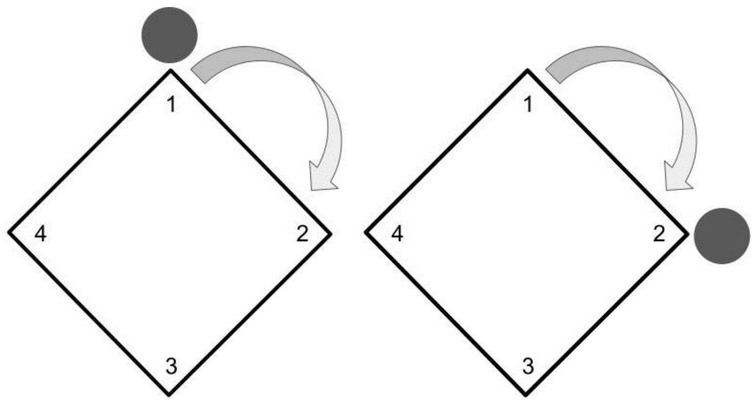

Alpha then plays $r^2$ to put the token back on 1.



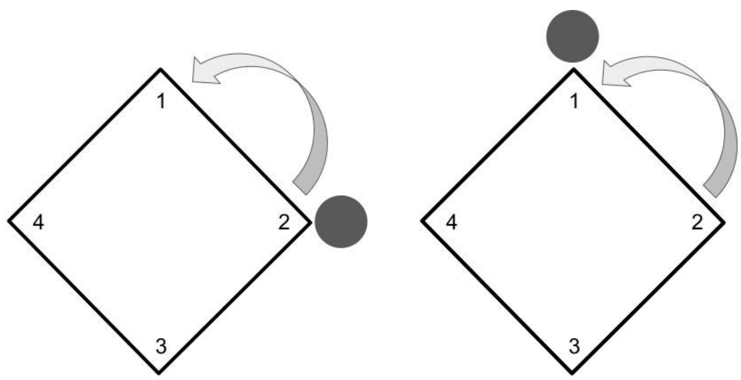

Since $r^2$ is its own inverse, Beta chooses $e$, since $e$ is also its own inverse.



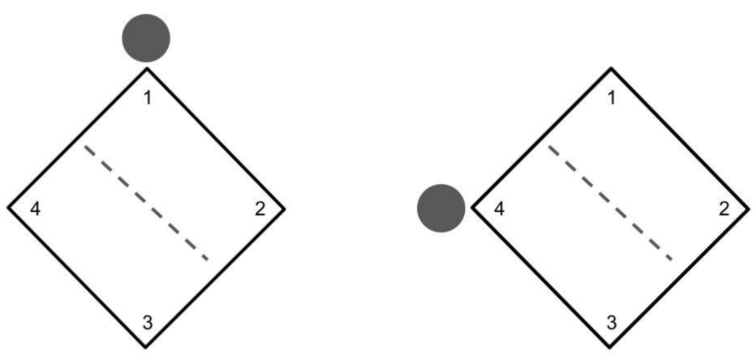

The product of those two moves is $r^2$. So the product of the moves played so far is $r^2 r^2 = e$. Alpha then plays $r^1$.



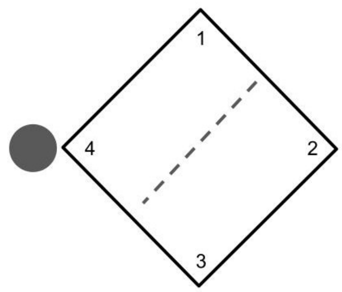

Beta chooses $r^3$, since $r^3 = r^{-1}$, the inverse of $r^1$, which puts the token back on 1.

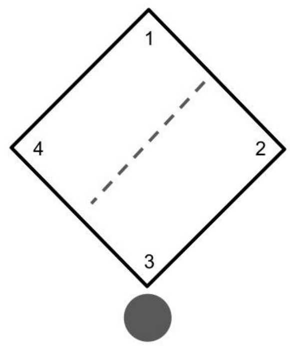



Alpha plays $f_{2.5}$, moving the token to Vertex 4.




Beta plays $f_{1.5}$, moving the token to Vertex 3. So the product of the moves played so far is $e * r^2 = e$.

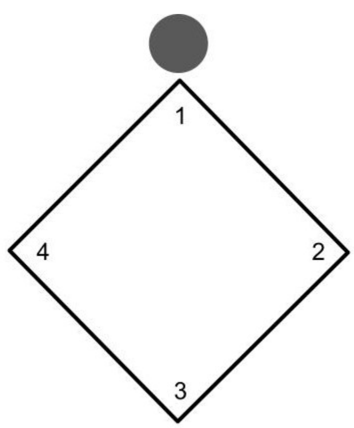




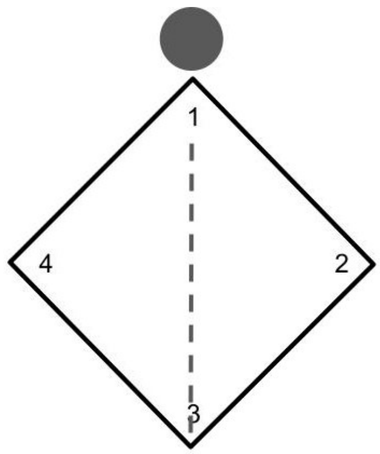

Beta won. Notice how the product of the pairs of moves that Beta picked were all $r^{n/2}$ or $e$. Since there were an odd number of 180° rotations, the token did not land back on 1. We'll show later that for an even $n$, we can always pair up the flips and rotations of $D_n$ in that way.

## 3.4 Square and Split Pair

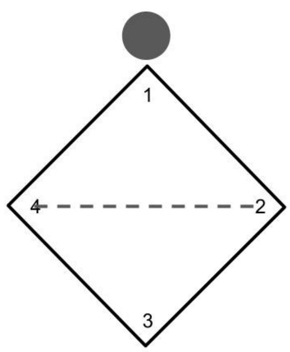
Suppose Alpha wanted to use that pairing strategy for the Move variation. What might that look like? Alpha has to go first, so Alpha chooses $e$. Before, $e$ was paired up with $r^2$. If Beta doesn't pick $r^2$ next, this pair will be split apart, with other moves in between them. We can refer to this idea as a split pair.

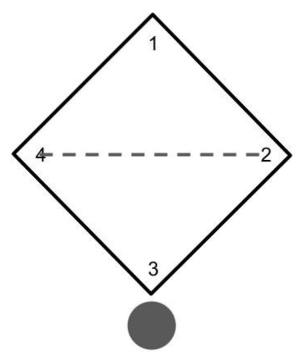



Suppose Beta chose $f_1$.

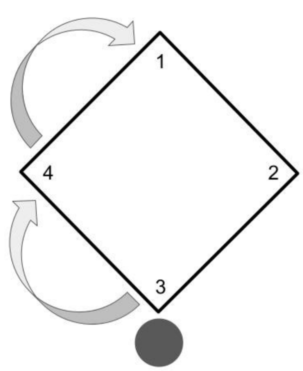


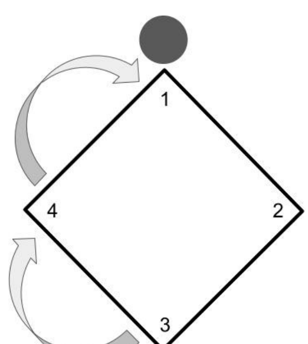

Alpha chooses $f_2$ to fit with their strategy, putting the token on Vertex 3.



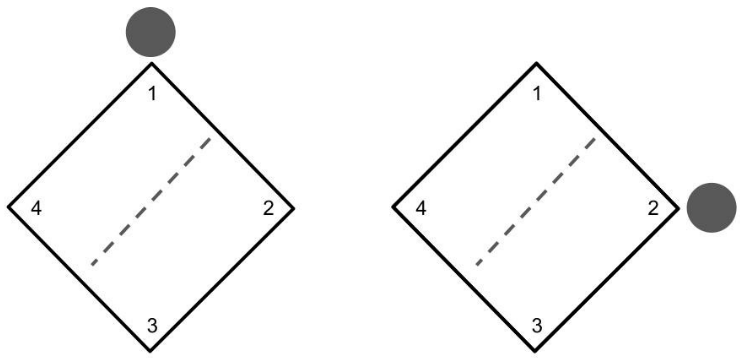




This means that the product of moves played so far is $e * r^2 = r^2$. Beta plays $r^2$, which has no inverse.

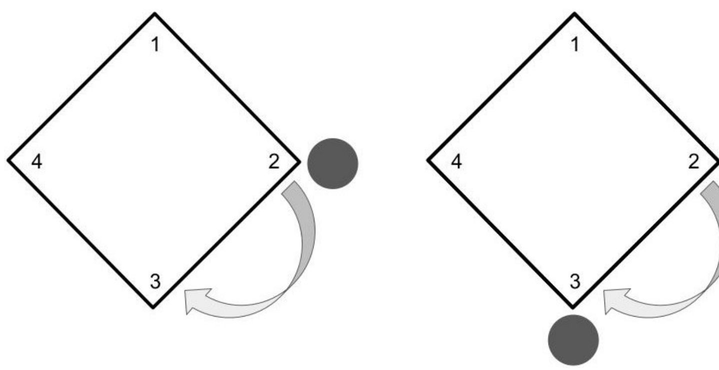


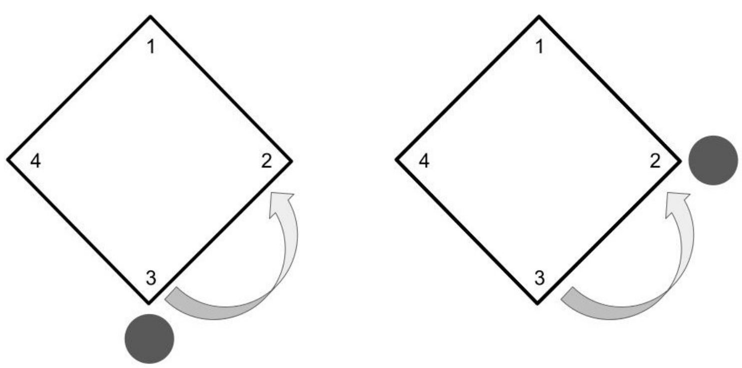
After all, the pair to $r^2$ was $e$, and Alpha already played $e$. Since they have to pick something, Alpha picks $f_{1.5}$, which puts the token on vertex 2. This means that unless Beta plays $f_{2.5}$ right now, the pair $f_{1.5}$ and $f_{2.5}$ will be another split pair.



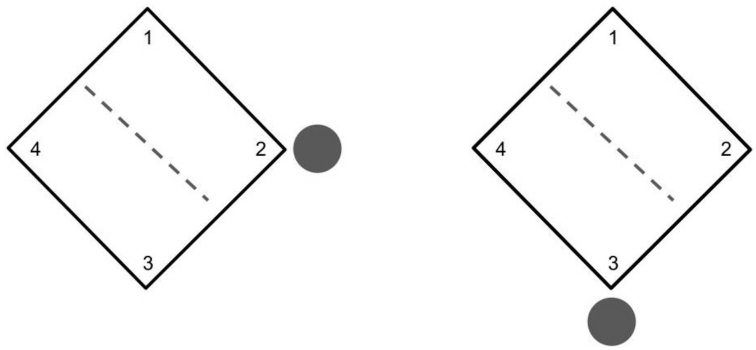

The product of moves played so far is $r^2 * r^2 * f_{1.5} = f_{1.5}$. Beta then plays $r^1$, putting the token on Vertex 3.




Alpha chooses $r^3$, since $r^3 = r^{-1}$, the inverse of $r^1$, which puts the token back on 2.

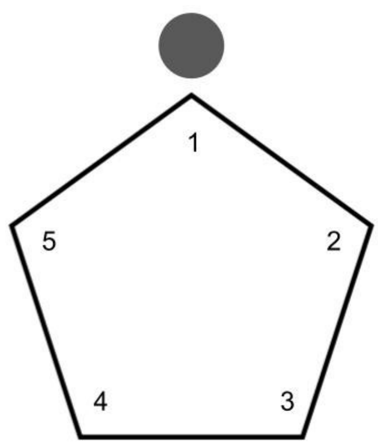


The product of moves played so far is $r^2 * r^2 * f_{1.5} * e$. Beta plays the final move, $f_{2.5}$, moving the token to Vertex 3.

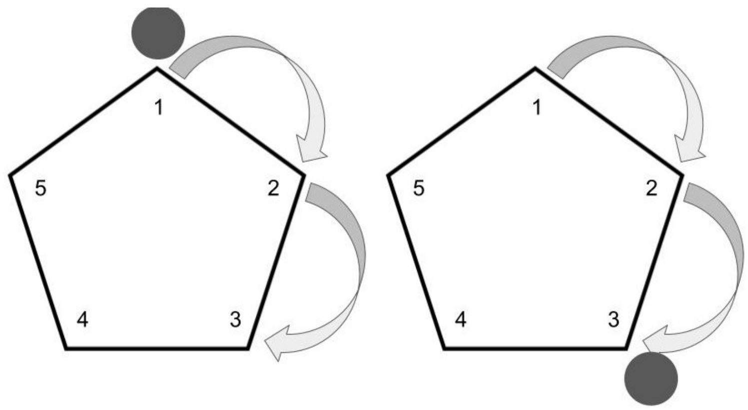



The product of moves played at the end was $r^2 * r^2 * f_{1.5} * e * f_{2.5} = r^2$. Since the token didn't land on Vertex 1, Alpha won. Even though the pairs $e$ and $r^2$ and $f_{1.5}$ and $f_{2.5}$ were split up and had some rotations in between, the final game still ended up the same. We'll prove later that this idea of ignoring split pairs works in general.

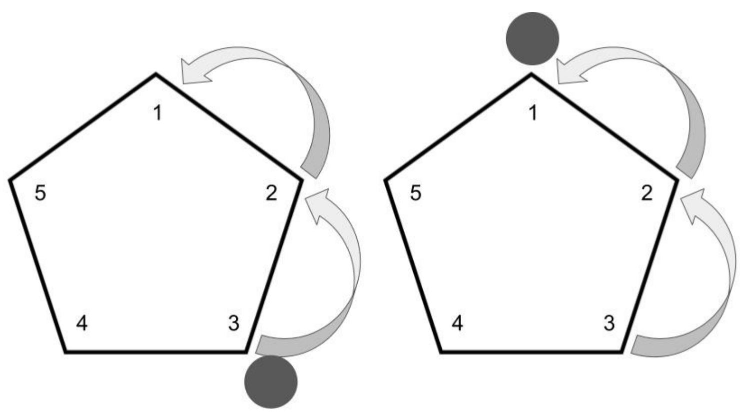
## 3.5 Example of a pentagon



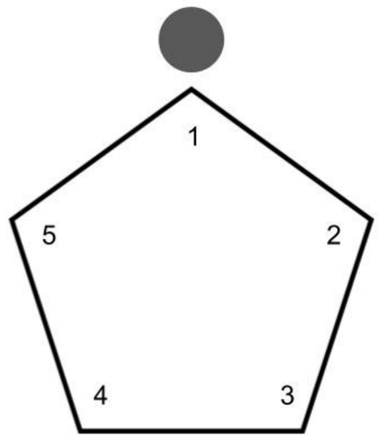

For the Return game, Beta remembers that pairing up flips and rotations has worked out well before, so Beta plans to pair up the flips and rotations just like they did for the triangle. So Beta decides to pair up rotations with their inverses, $f_1$ with $e$, $f_2$ with $f_3$ and $f_4$ with $f_5$. We'll see why keeping the same strategy for the triangle will not work for the pentagon. Since we're pairing moves up, we can look at the resultant moves of their pairs, as opposed to individual moves. Alpha starts with $f_2$, so Beta plays $f_3$, giving a resultant move of $r^2$. We'll see later that this was a bad move on Beta's part.



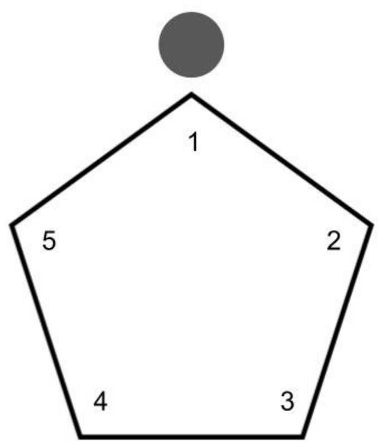

Alpha plays with $f_5$, so Beta plays $f_4$, giving a resultant move of $r^{-2} = r^3$.



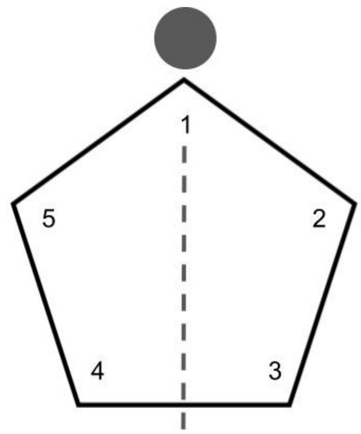

Alpha plays $r^2$, which Beta pairs with $r^{-2} = r^3$ to get a resultant move of $e$. The token is still on 1.




Alpha plays $r^4$, which Beta pairs with $r^{-4} = r^1$ to get a resultant move of $e$.



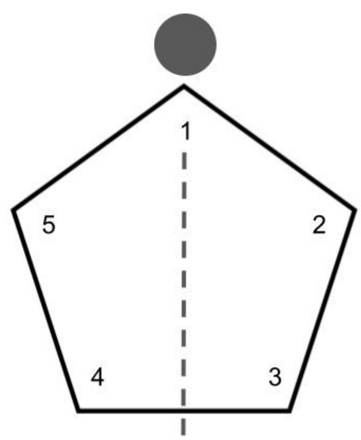

Finally, Alpha plays $f_1$, so Beta plays $e$. The resultant move is $f_1$.



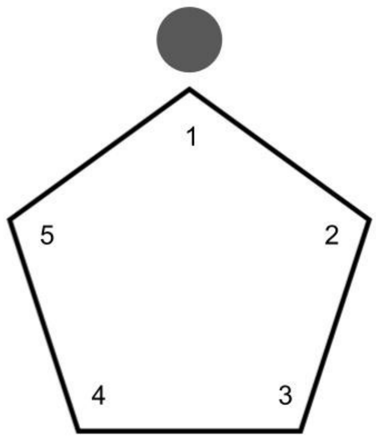

The product of all the moves was $r^2 r^{-2} f_1 = f_1$. The token ends on Vertex 1, so Alpha wins Return. Because the resultant moves of Beta's pairs for the flips were an even number of rotations $r^{\pm 2}$, Alpha was able to get the final product to one which fixed Vertex 1.

## 3.6 Pentagon, take 2

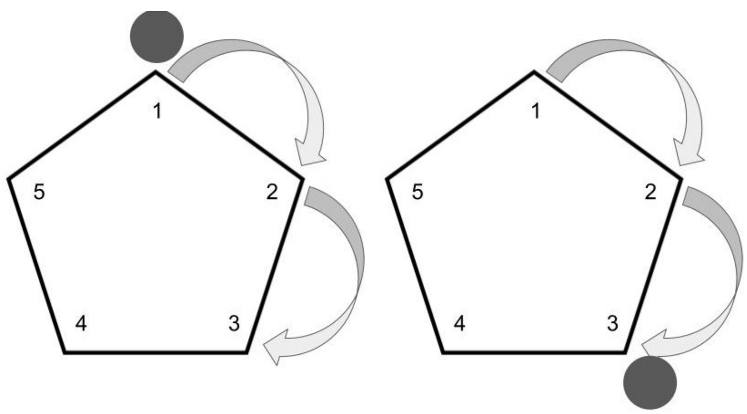
Let's try Return again. Since that strategy didn't work, Beta decides to pair up $f_2$ with $f_5$ and $f_3$ with $f_4$ instead. We'll show later that this is a much better plan. Alpha plays $f_1$, so Beta plays $e$. The resultant move is $f_1$.



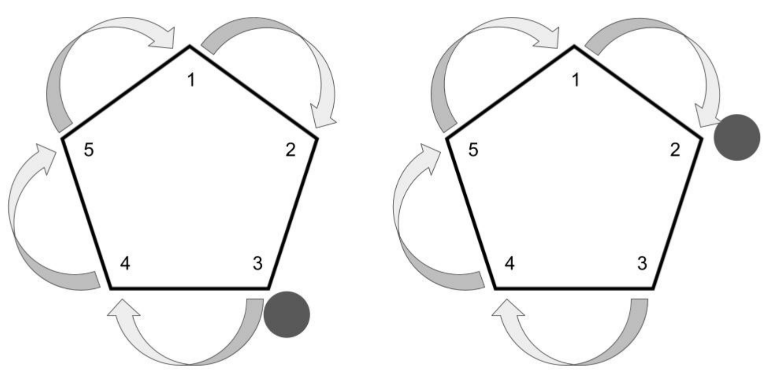

Alpha plays $r^2$, so Beta plays $r^{-2} = r^3$. The resultant move is $e$.



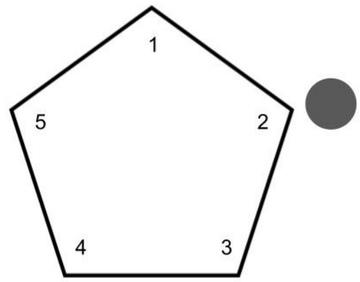

Alpha plays $f_3$ and Beta plays $f_4$ the resultant move is $r^2$.




So the product of moves played so far is $f_1r^2$. Alpha plays $f_5$ and Beta plays $f_2$. The resultant move is $r^4$.

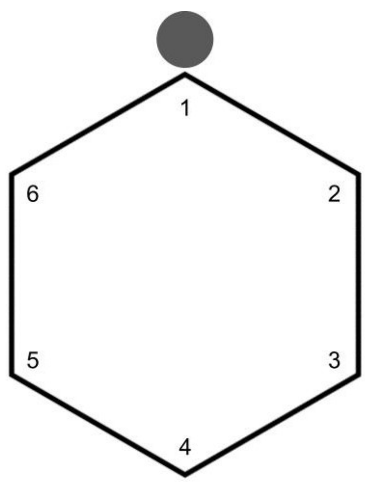


Alpha plays $r^4$, so Beta plays $r^{-4} = r^1$. The resultant move is $e$, leaving the token on Vertex 2.

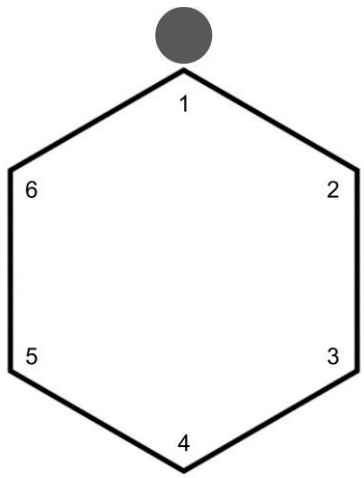



The product of all the moves at the end was $f_1 r^2 r^4 = f_1 r^2$ The difference between this example and the previous example was Beta's pairing strategy. Where before, we had one rotation of $r^2$ and one of $r^{-2}$, this time, we had one rotation of $r^2$ and one of $r^4$. No combination of $\pm 2 \pm 4$ will equal 0, so we'll show later that Alpha can't win when Beta uses this strategy.

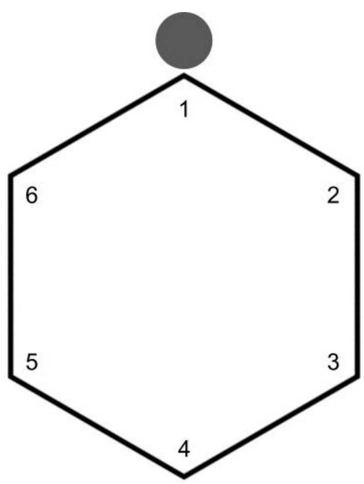
## 3.7 Example of a hexagon



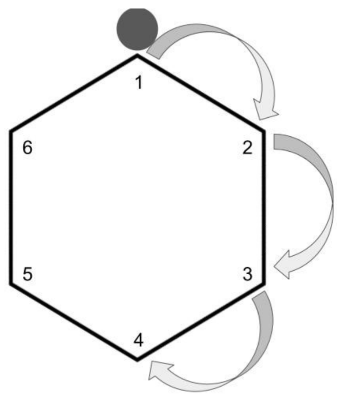

For the return game, Alpha decides to pair up all the moves to give them all resultant moves of the 180° rotation or $e$. Note that this is the same strategy that was used for the square. If this works, this would be the first strategy that lets Alpha win the Return game. Alpha starts with $e$, leaving the token on 1.

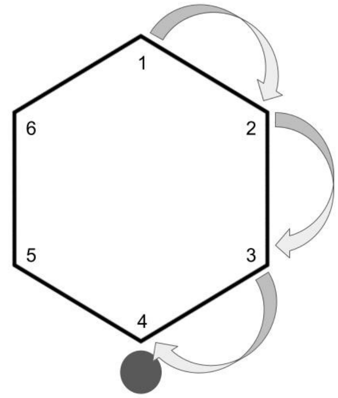



Beta plays $r^2$, so Alpha pairs that with $r^{-2} = r^4$ to get a resultant move of $e$.

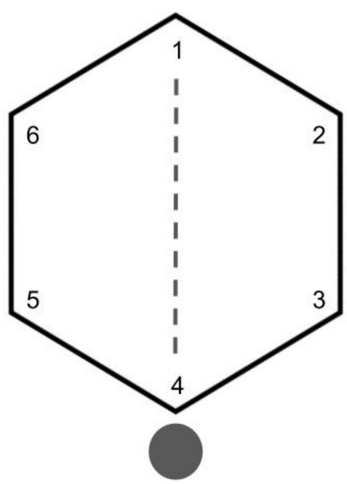



Beta plays $r^3$.

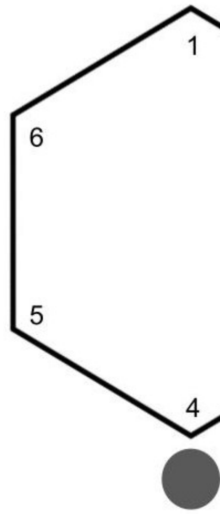




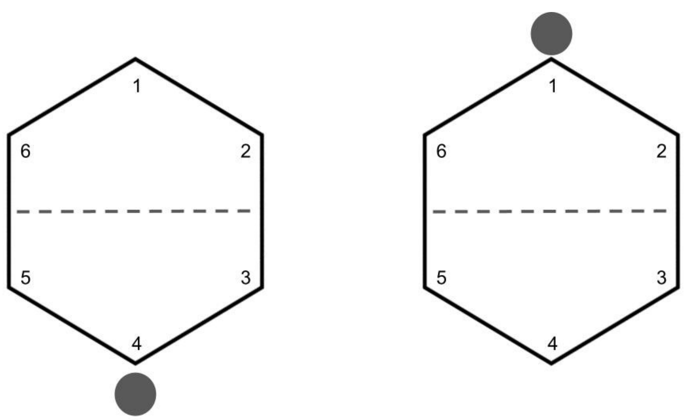

But $r^3$ is its own inverse, so Alpha picks $f_1$. The product of moves this far is $r^3 f_1$.



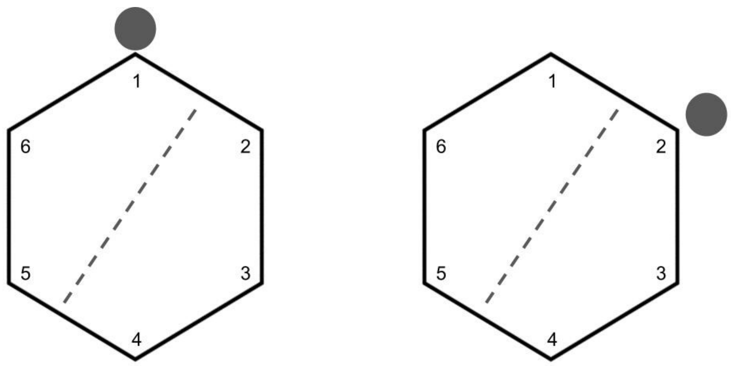

Beta plays $r^1$, so Alpha pairs that with $r^{-1} = r^5$ to get a resultant move of $e$.



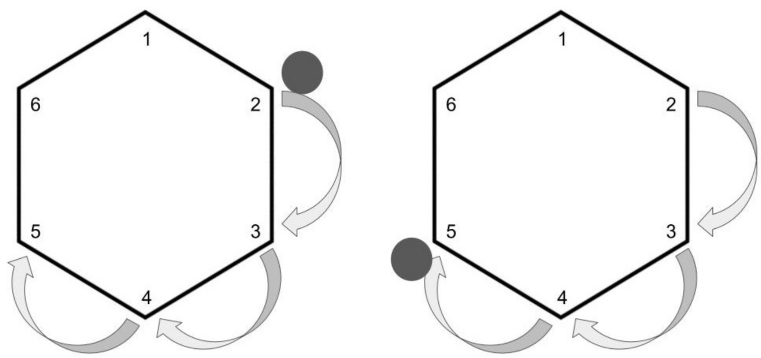

Beta plays $f_{2.5}$, but Alpha already played $f_{2.5}$'s pair. This completes that "split pair".



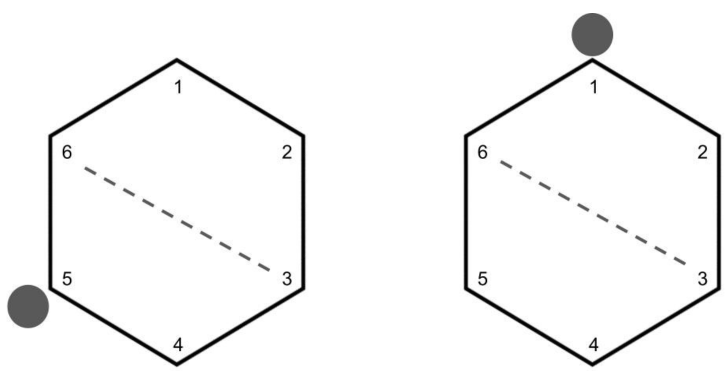

The product of moves this far is $r^3 f_1 f_{2.5} = r^3 r^3 = e$. So Alpha plays $f_{1.5}$.




Beta plays $f_{3.5}$, so Alpha plays $f_2$ to give a resultant move of $r^3$. So the product of moves so far is $f_{1.5}r^3$.




The only move left is for Beta to play $f_3$, the pair to $f_{1.5}$.




The product of moves at the end is $f_{1.5}r^3f_3$. Since $r^3$ is in the center, $f_{1.5}r^3f_3 = r^3f_{1.5}f_3 = e$. Even with all the split pairs, Alpha's strategy worked out to be four rotations by 3, bringing us back to

1, which means that Alpha won Return by pairing up flips and rotations to give us an even number of rotations of $r^{n/2}$. We'll show in the next chapter that this strategy will always work out.

# Chapter 4

# Formal Results

## 4.1 General Strategy

Now we can determine who will win for any $D_n$, and more interestingly, why. We begin with a general strategy. We want to show that either player can make the product of the resultant moves of their pairing strategy

$$g_{|G|} = (r^{n/2})^{n/2+1}$$

if $n$ is even, and

$$g_{|G|} = f_1 r^M$$

if $n$ is odd, where specifically $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$ for some $b_1 \in \{\pm b\}$ and $a_i \in \{\pm 2\}$. We do this by a paring strategy, like those seen in the examples. For any n, we start by pairing up inverse rotations to get a resultant move of $e$. For the evens, we pair all remaining moves such that the resultant move is always $r^{n/2}$. This results in exactly $n/2$ rotations of $r^{n/2}$ from the flips and one rotation of $r^{n/2}$ from pairing $e$ with $r^{n/2}$. For the odds, we pair flips in such a way that we get either an odd number of rotations $r^{\pm 2}$ or an odd number of rotations $r^{\pm 2}$ and one rotation $r^{\pm 4}$. First, we demonstrate that $g_{|G|}$ will equal a permutation of the product of pairs, by showing that there exists a strategy for either player to pair up the elements as required. An important part of the lemma is showing that regardless of if Player has to go first, in a sense creating the split pairs we saw in the examples, Player can still force the product to be $g_{|G|} = (r^{n/2})^{n/2+1}$ if $n$ is even, and $g_{|G|} = f_1 r^M$ if $n$ is odd. Finally we show that regardless of the order in which the pairs are played, the product is still of the form $g_{|G|} = (r^{n/2})^{n/2+1}$ if $n$ is even, and $g_{|G|} = f_1 r^M$ if $n$ is odd.

**Lemma 6.** *Let $n > 1$, $G = D_n$, $\mathcal{F}$ denote the set of flips, and $\mathcal{F}' = \mathcal{F} \cup \{e, r^{n/2}\}$ if $n$ is even and $\mathcal{F}' = \mathcal{F} \cup \{e\}$ if $n$ is odd. Let $a$ and $b$ be integers such that $a = n/2 = b$ if $n$ is even and $a = 2$ and $b \in \{2, 4\}$ if $n$ is odd. Suppose that $\mathcal{F}'$ can be partitioned into sets $\{x_i, x_i'\}$ of size 2 for $0 \le i \le |\mathcal{F}'|/2 - 1$ such that*

1. *$\{x_0, x_0'\}$ is $\{e, f_1\}$ if $n$ is odd and $\{e, r^{n/2}\}$ if $n$ is even,*
2. *$x_1 x_1' \in \{r^{\pm b}\}$, and*
3. *$x_i x_i' \in \{r^{\pm a}\}$ for $2 \le i \le \frac{|\mathcal{F}'|}{2} - 1$,*

*Then either player can ensure that the game ends with*

$$g_{|G|} = f_1^F r^M$$

*where $F = 1$ and $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$ for some $b_1 \in \{\pm b\}$ and $a_i \in \{\pm 2\}$ if $n$ is odd and $F = 0$ and $M = (n/2)^{n/2+1}$ if $n$ is even.*

*Proof.* Let $n > 1$, $G = D_n$, $\mathcal{F}$ denote the set of flips, and $\mathcal{F}' = \mathcal{F} \cup \{e, r^{n/2}\}$ if $n$ is even and $\mathcal{F}' = \mathcal{F} \cup \{e\}$ if $n$ is odd. Let $a$ and $b$ be integers such that $a = n/2 = b$ if $n$ is even and $a = 2$ and $b \in \{2, 4\}$ if $n$ is odd. Suppose that $\mathcal{F}'$ can be partitioned into sets $\{x_i, x_i'\}$ of size 2 for $0 \le i \le |\mathcal{F}'|/2 - 1$ such that

1. $\{x_0, x_0'\}$ is $\{e, f_1\}$ if $n$ is odd and $\{e, r^{n/2}\}$ if $n$ is even,
2. $x_1 x_1' \in \{r^{\pm b}\}$, and
3. $x_i x_i' \in \{r^{\pm a}\}$ for $2 \le i \le \frac{|\mathcal{F}'|}{2} - 1$,

Let Player be the player who will make $g_{|G|} = f_1^F r^M$. Let Opponent be the other player. Let $K = |\mathcal{F}'|/2$.

First, we determine if there is a strategy for each player to make the game end with $g_{|G|} = \prod_{i=0}^{K-1} y_{\sigma(i)}$ where $y_{\sigma(i)} \in \{x_i x_i', x_i' x_i\}$. This means that we want to show that there is a strategy for either player to make the game end as some permutation, $\sigma$, of the product of pairs $x_i$ and $x_i'$. Note that for some $\sigma$, $y_\sigma(i) = x_i x_i'$ or $y_\sigma(i) = x_i' x_i$.

We will denote each player's choice as $h_i$ on the $i$th term. So $g_{|G|} = \prod_{i=1}^{2n-1} h_i hi + 1$ If Player is Alpha, the game starts with Player defining $h_1 := r^{n/2}$ if $n$ is even and $h_1 := f_1$ if $n$ is odd. Following this, Player follows the following strategy after Opponent defines $h_i$, where $X_i$ is defined to be $\{h_1, \dots, h_i\}$, the set of previously-played elements. To account for split pairs, define $k_0 = 1$, which will be used in the recursive definition below. We recursively do the following strategy after $k_\ell$ has been defined, such that $\ell$ is the number of times a pair has been split at that point and $h_{k\ell}$ is the move when the split pair is played.

1. If Opponent defines $h_i = r^j$ for some $j \neq e, n/2$, then Player defines $h_{i+1} := r^{-j}$. In this case, $h_i h_{i+1} = e$.
2. If Opponent defines $h_i \in \{x_j, x'_j\}$ for some $j \neq k_\ell$, then Player defines $h_{i+1}$ to be the element in $\{x_j, x'_j\} \setminus \{h_i\}$. In this case, $h_i h_{i+1} \in \{r^{\pm 2}, r^{\pm b}\}$.
3. When Opponent defines $h_i$ to be in $\{x_{k_\ell}, x'_{k_\ell}\}$, then $h_i$ is the pair to a move already played. If $\mathcal{F}' \not\subseteq X_i$ then a flip remains, and so Player defines $h_{i+1} = x_j$, where $j$ is the smallest subscript such that $x_j \notin X_i$. If $\mathcal{F}' \subseteq X_i$, then no flips remain and Player defines $h_{i+1}$ to be $r^j$, where $j$ is the smallest positive integer such that $r^j \notin X_i$. Then we define $k_{\ell+1} = i + 1$.

This continues until the elements are exhausted.

If Player is Beta, then $k_1$ is never defined, and we have

$$g_{|G|} = \prod_{i=1}^{n} h_{2i-1} h_{2i},$$

where $h_{2i-i} h_{2i} = e$ if $h_{2i-1}$ is a rotation other than $e$ or $r^{n/2}$, and is in $\{x_j x'_j, x'_j x_j\}$ for some $j$ otherwise. This is sufficient to conclude that $g_{|G|} = \prod_{i=0}^{K-1} y_{\sigma(i)}$ for some permutation $\sigma$ and $y_j \in \{x_j x'_j, x'_j x_j\}$.

If Player is Alpha, we get a sequence $k_0, k_1, \dots, k_\ell$ defined from above for some $\ell$, the number of times a pair was split, such that the game is the product of products of split pairs $\{x_{k_j}, x'_{k_j}$ with the product of $m_j$ pairs between them.

So

$$g_{|G|} = \prod_{j=0}^{\ell-1} h_{k2j} (\prod_{i=m_0}^{m_j} h_{2i} h_{2i+1}) h_{k2} j + 1$$

and since each product of pairs is a rotation, $(\prod_{i=m_0}^{m_j} h_{2i}h_{2i+1}) = r^{\pm c_j}$, we see that

$$g_{|G|} = \prod_{j=0}^{\ell-1} h_{kj} r^{\pm c_j} h_{2i}h_{2i+1}) = \prod_{j=0}^{\ell-1} h_{kj} h_{2i}h_{2i+1}) r^{\pm c_j}$$

and likewise,

$$g_{|G|} = (\prod_{j=0}^{\ell-1} h_{k_{2j}} h_{k_2j+1} ( \prod_{i=m_0}^{m_j} h_{2i}h_{2i+1}))$$

Thus, $g_{|G|}$ is a permutation of the product of pairs and so

$$g_{|G|} = \prod_{j=0}^{\ell-1} y_{\sigma(i)}$$

for some permutation $\sigma$ and $y_j \in \{x_j x_j', x_j' x_j\}$.

We will now demonstrate that $\prod_{i=0}^{K-1} y_{\sigma(i)}$ for some permutation $\sigma$ of $\{0, 1, 2, \ldots, K-1\}$, where $y_{\sigma(i)} \in \{x_i x_i', x_i' x_i\} = f_1^F r^M$, where $F = 1$ and $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$ for some $b_1 \in \{\pm b\}$ and $a_i \in \{\pm 2\}$ if $n$ is odd and $F = 0$ and $M = n/2)^{n/2+1}$ if $n$ is even. This means that regardless of the permutation, the product of pairs will equal $g_{|G|} = f_1^F r^M$. First, if $n$ is even, then all $y_i = r^{n/2}$, which is in the center, so will yield the desired result of $g_{|G|} = (r^{n/2})^{n/2+1}$, since $K = |\mathcal{F}'|/2 = (n+2)/2 = n/2+1$ in this case. If $n$ is odd, we have that $\prod_{i=0}^{K-1} y_{\sigma(i)}$ is a product of $y_{\sigma(i)}$. Let $m$ be such that $\sigma(m) = 0$. Then $y_{\sigma(m)} = f_1$, and

$$g_{|G|} = \prod_{i=0}^{K-1} y_{\sigma(i)} = \left( \prod_{i=0}^{m-1} y_{\sigma(i)} \right) (y_0) \left( \prod_{i=m+1}^{K-1} y_{\sigma(i)} \right).$$

Since all $y_{\sigma(i)}$ for $i \neq m$ are rotations, $\prod_{i=0}^{m-1} y_{\sigma(i)} = r^d$ for some $d$, so

$$
\begin{aligned}
g_{|G|} &= \prod_{i=0}^{K-1} y_{\sigma(i)} \\
&= \left(\prod_{i=0}^{m-1} y_{\sigma(i)}\right)(y_0)\left(\prod_{i=m+1}^{K-1} y_{\sigma(i)}\right) \\
&= \left(\prod_{i=0}^{m-1} y_{\sigma(i)}\right)(f_1)\left(\prod_{i=m+1}^{K-1} y_{\sigma(i)}\right) \\
&= r^d f_1 \left(\prod_{i=m+1}^{K-1} y_{\sigma(i)}\right) \\
&= f_1 r^{-d} \left(\prod_{i=m+1}^{K-1} y_{\sigma(i)}\right) \\
&= f_1 \left(\prod_{i=0}^{m-1} y_{\sigma(i)}^{-1}\right)\left(\prod_{i=m+1}^{K-1} y_{\sigma(i)}\right),
\end{aligned}
$$

Since $y_i \in \{r^{\pm 2}, r^{\pm b}\}$ for $i \neq 0$, we conclude that $y_{\sigma(i)}^{-1} \in \{r^{\pm 2}, r^{\pm b}\}$ for $i \neq m$. Since there is exactly one $y_i \in \{r^{\pm b}\}$ with the others in $\{r^{\pm 2}\}$, we conclude that

$$g_{|G|} = f_1 r^{b_1 + \sum_{i=2}^{K} a_i}$$

in this case, where $b_1 \in \{\pm b\}$ and each $a_i \in \{\pm 2\}$.

Then either player can ensure that the game ends with

$$g_{|G|} = f_1^F r^M$$

where $F = 1$ and $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$ for some $b_1 \in \{\pm b\}$ and $a_i \in \{\pm 2\}$ if $n$ is odd and $F = 0$ and $M = n/2)^{n/2+1}$ if $n$ is even.

□

**Corollary 7.** *If Player has a winning strategy for Return, then Opponent has a winning strategy for Move.*

*Proof.* Suppose Player has a winning strategy for Return such that Player can make $g_{|G|} = (f_1)^F r^M$. Then Opponent can also make $g_{|G|} = (f_1)^F r^M$. If $1^{g_{|G|}} = 1^{(f_1)^F r^M} = 1$, Alpha wins Return and

Beta wins Move. If $1^{g_{|G|}} = 1^{(f_1)^F r^M} \neq 1$, then Alpha wins Move and Beta wins Return. $\square$

These two lemmas can give us the results we need for each $D_n$.

## 4.2 Results

**Theorem 8.** *Let $G = D_{2n}$, and suppose that $n \equiv 3 \mod 4$. Then Beta has the winning strategy for Return and Alpha has a winning strategy for Move.*

*Proof.* Let $\mathcal{P}$ denote $\{\{e, f_1\}\} \cup \{\{f_{2j}, f_{2j+1}\} \mid 2 \leq j \leq \frac{n-1}{2}\}$. By Lemma 2, $\mathcal{P}$ partitions the set of flips with the identity such that $x_i x_i' = r^2$ for all $\{x_i, x_i'\} \in \mathcal{P}$, provided $\{x_i, x_i'\} \neq \{e, f_1\}$. Thus, we may let $a = 2 = b$ in the language of Lemma 6 to conclude that Beta ensures that the final element is $f_1 r^M$ for some $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$, where $b_1, a_i \in \{\pm 2\}$.

We will now demonstrate that $M \not\equiv 0 \mod n$. First, note that

$$|M| \leq 2 + ((n-3)/2)(2) = 2 + (n-3) = n - 1 < n,$$

so $r^M$ can only be trivial if $M = 0$. However, $M$ is a sum of $1 + (n-3)/2$ terms, all of which are $\pm 2$. Since $n \equiv 3 \mod 4$, there are an odd number of terms, so there cannot be an equal number of positive terms and negative terms. Therefore, $M \neq 0$, and $r^M$ is a nontrivial rotation. Then $1^{g_{|G|}} = 1^{f_1 r^M} = (1^{f_1})^{r^M} = 1^{r^M} \neq 1$, so Beta has a winning strategy for Return if $n \equiv 3 \mod 4$. Since Beta has a winning strategy for Return, Lemma 7 tells us that Alpha has a winning strategy for Move.

$\square$

**Theorem 9.** *Let $G = D_{2n}$, and suppose that $n \equiv 1 \mod 4$. Then Beta has the winning strategy for Return and Alpha has a winning strategy for Move.*

*Proof.* Let $G = D_{2n}$, and suppose that $n \equiv 1 \mod 4$. Let $\mathcal{P}$ denote

$$\{\{e, f_1\}, \{f_2, f_n\}\} \cup \{\{f_{2j-1}, f_{2j}\} \mid 2 \leq j \leq \frac{n-1}{2}\}.$$

By Lemma 2, $\mathcal{P}$ partitions the set of flips with the identity such that $x_i x_i' = r^2$ for all $\{x_i, x_i'\} \in \mathcal{P}$, provided $\{x_i, x_i'\} \neq \{e, f_1, f_2, f_n\}$.

Thus, we may let $a = 2$ and $b = 4$ in the language of Lemma 6 to conclude that Beta ensures that the final element is $f_1 r^M$ for some $M = b_1 + \sum_{i=1}^{(n-3)/2} a_i$, where $b_1 = \pm 4$ and $a_i \in \{\pm 2\}$. We will now demonstrate that $M \not\equiv 0 \mod n$. Since

$$|M| \leq 4 + ((n-3)/2)(2) = 4 + (n-3) = n + 1 < 2n,$$

there are only three possible values for $M$ such that $M \equiv 0 \mod n$, those being $M = \pm n$ or $M = 0$. First, note that $M$ is the sum of even numbers, so $M$ is even and thus, $M \neq \pm n$ , thus, $r^M$ can only be trivial if $M = 0$. However, since $(n-3)/2$ is odd, $\sum_{i=1}^{(n-3)/2} a_i \neq \pm 4$, Therefore, $M \neq 0$, and $r^M$ is a nontrivial rotation. Then $1^{g_{|G|}} = 1^{f_1 r^M} = (1^{f_1})^{r^M} = 1^{r^M} \neq 1$, so Beta has a winning strategy for Return if $n \equiv 1 \mod 4$. Since Beta has a winning strategy for Return, Lemma 7 tells us that Alpha has a winning strategy for Move. $\square$

**Theorem 10.** *Let $G = D_{2n}$, and suppose that $n \equiv 0 \mod 4$. Then Beta has the winning strategy for Return and Alpha has a winning strategy for Move.*

*Proof.* Let $G = D_{2n}$, and suppose that $n \equiv 0 \mod 4$. Let $\mathcal{P}$ denote

$$\{\{e, r^{n/2}\}, \{f_j, f_{j+n/4}\}\} \mid 1 \leq j \leq \frac{n-1}{4}\}.$$

Lemma 6 tells us that Beta can ensure that $g_{|G|} = r^{n/2}$ when $n/2$ is even. Then $1^{g_{|G|}} = 1^{r^{n/2}} \neq 1$, so Beta has a winning strategy for Return if $n \equiv 0 \mod 4$. Since Beta has a winning strategy for Return, Lemma 7 tells us that Alpha has a winning strategy for Move. $\square$

**Theorem 11.** *Let $G = D_{2n}$, and suppose that $n \equiv 2 \mod 4$. Then Alpha has the winning strategy for Return and Beta has a winning strategy for Move.*

*Proof.* Let $G = D_{2n}$, and suppose that $n \equiv 2 \mod 4$. Let $\mathcal{P}$ denote

$$\{\{e, r^{n/2}\}, \{f_j, f_{j+n/4}\}\} \mid 1 \leq j \leq \frac{n-1}{4}\}.$$

By Lemma 6, we have that $g_{|G|} = r^{n/2(n/2)+1}$. Since $(n/2)+1$ is even, this is to say that $g_{|G|} = e$, which fixes 1. Alpha has a winning strategy. Then $1^{g_{|G|}} = 1^e = 1$, so Alpha has a winning strategy for Return if $n \equiv 2 \mod 4$. Since Alpha has a winning strategy for Return, Lemma 7 tells us that Beta has a winning strategy for Move. $\square$

# Chapter 5

# Future Research

The next step for this work is to examine other nonabelian groups. For example, we could examine the quaternion group, $Q_8$. The quaternion group is a group with 8 elements, $Q_8 = \{e, i, j, k, -e, -i, -j, -k\}$ such that $e$ is the identity and $(-e)^2 = e$, $i^2 = j^2 = k^2 = ijk = -e$. In general, looking at families of permutation groups, like symmetric groups, and alternating groups would be a logical next step for this project.

Besides looking at different groups, we could also consider alterations to the rules of the game, for example, where elements of $G$ can be played more than once, the goals of the players change from landing the token on Vertex 1 to landing the token on any even Vertex, or introducing a third player.

# Games on Dihedral Groups

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