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Anne Sinko
College of Saint Benedict/Saint John's University, asinko@csbsju.edu

Peter J. Slater

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Efficient Domination in Knights Graphs

Anne Sinko * and Peter J. Slater +

* Mathematical Sciences Department

+ Computer Science Department

University of Alabama in Huntsville

Huntsville, AL 35899 USA

E-mail: *philliac@email.uah.edu* , *slater@math.uah.edu*

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Abstract

The influence of a vertex set $S \subseteq V(G)$ is $I(S) = \sum_{v \in S}(1 + deg(v)) = \sum_{v \in S} |N[v]|$, which is the total amount of domination done by the vertices in S. The efficient domination number $F(G)$ of a graph G is equal to the maximum influence of a packing, that is, $F(G)$ is the maximum number of vertices one can dominate under the restriction that no vertex gets dominated more than once.

In this paper, we consider the efficient domination number of some finite and infinite knights chessboard graphs.

Keywords: efficient domination number,knights graphs

2000 Mathematics Subject Classification: 05C69

1. Introduction

Corresponding to the chess pieces queen, rook, bishop, knight, and king there are graphs $Q_{j,k}, R_{j,k}, KN_{j,k}$, and $KI_{j,k}$, each of order $n = jk$, where the vertex set corresponds to the jk squares of a j-by-k board, and two vertices are adjacent if and only if the given chess piece can go from one of the two vertices' corresponding squares to the other corresponding square in one move. In this paper we let $v_{i,j}$ or simply (i,j) denote the vertex in the i th row and j th column. For example, in $Q_{8,8}$ the vertex $v_{3,2} = (3,2)$ has the closed neighborhood $N[v_{3,2}] = \{v_{1,2}, v_{2,2}, v_{3,2}, v_{4,2}, \ldots, v_{8,2}, v_{3,1}, v_{3,3}, \ldots, v_{3,8}, v_{2,1}, v_{4,3}, v_{5,4}, \ldots, v_{8,7}, v_{4,1}, v_{2,3}, v_{1,4}\}$ with cardinality $|N[v_{3,2}]| = 1 + deg(v_{3,2}) = 24$. As noted in Sinko and Slater [8], (upper and lower) independence, domination and irredundance parameters for these graphs have been extensively studied. For an excellent survey see Hedetniemi, Hedetniemi, and Reynolds [7]. Another wonderful exposition concerning these parameters is given by Watkins in [11].

In [8], we introduced the study of influence parameters for chessboard graphs, namely, efficient domination F, closed neighborhood order domination W, closed neighborhood order packing P, and redundance R, along with the linear programming versions of the parameters, $F_f = W_f$ and $P_f = R_f$. See, for example, Grinstead and Slater [5] and Slater [9]. In this paper, we consider the efficient domination number of both finite and infinite knights chessboard graphs.

Unlike the cases where one seeks, for example, a vertex set S in a graph $G = (V, E)$ that is a packing, independent set, dominating set, etc. and is interested in the cardinality of S, for the influence parameters F and R one is interested in how much domination is done by S. Because each vertex v in S dominates precisely the vertices in its closed neighborhood $N[v]$, the *influence* of S is defined in [5] to be $I(S) = \sum_{v \in S} |N[v]| = \sum_{v \in S}(1 + deg(v))$, where $deg(v) = |N(v)|$, the cardinality of the open neighborhood of v.

A vertex set S is called a perfect code (see [4]) or an efficient domination set (see [2, 3]) for a graph G if every vertex in G is dominated exactly once, that is, for each $v \in V(G)$ we have $|N[v] \cap S| = 1$. Not every graph G has an efficient dominating set, and the efficient domination number of G (see [1, 5, 6]) equals the maximum number of vertices that can be dominated by a vertex set S that does not dominate any vertex more than once. Because S must not dominate any vertex more than once, any two vertices u and v in S must have distance $d(u, v) \geq 3$, that is, S must be a packing. Thus, the *efficient domination number* of a graph G is $F(G) = max\{I(S) : S$ is a packing$\} = max\{\sum_{v \in S}(1 + deg(v)) : S \subseteq V(G)$ and $u, v \in S$ implies $d(u, v) \geq 3\}$. An $F(G)$-set S is a set that is both a packing and $I(S) = F(G)$. For definitions of parameters W, P, and R, one can see [8].

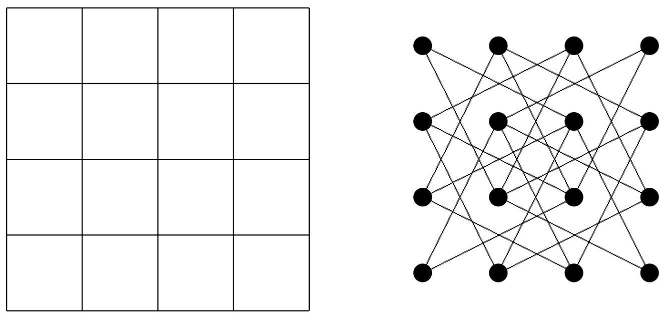

Figure 1: Two representations of $KN_{4,4}$.

Figure 1 illustrates the knights graph $KN_{4,4}$. Notice that each vertex corresponds to one of the squares of a j-by-k rectangular board. We consider the *coordinate neighbors* of a vertex v to be the vertices corresponding to the vertices/squares immediately to the left, right, below, or above the vertex/square corresponding to v. They are not adjacent to v in $KN_{j,k}$. The neighbors of $v_{a,b}$ are $\{v_{a\pm1,b\pm2}\} \cup \{v_{a\pm2,b\pm1}\}$. We let $v_{1,1}$ be the lower, left vertex/square. For the five-by-five knights graph $KN_{5,5}$ with $S = \{v_{1,1}, v_{3,3}, v_{3,4}\}$, as illustrated in Figure 2, we have $deg(v_{1,1}) = 2, deg(v_{3,3}) = 8$, and $deg(v_{3,4}) = 6$, and S is a packing with influence $I(S) =$

$(1+2)+(1+8)+(1+6) = 19$. One can show that $(F(KN_{1,1}), F(KN_{2,2}), \ldots, F(KN_{5,5})) = (1,4,7,12,19)$. The 19 vertices dominated by the $F(KN_{5,5})$-set $\{v_{1,1}, v_{3,3}, v_{3,4}\}$ are marked in Figure 2. A general formula for $F(KN_{k,k})$ is, at this time, unknown.

X	X		X	X
X		N		X
X	X	N		X
X	X	X	X	X
N	X		X	

Figure 2: $F(KN_{5,5}) = 19$

It is easy to see that $KN_{1,k}$ and $KN_{2,k}$ are efficiently dominatable (that is, each of these graphs has an efficient dominating set).

Theorem 1.1. [8] $F(KN_{1,k}) = k$ and $F(KN_{2,k}) = 2k$ for all $k \geq 1$.

Note that $KN_{3,3}$ consists of a cycle, C_8, and an isolated vertex corresponding to $v_{2,2}$, and $F(KN_{3,3}) = 7 < |V(KN_{3,3})| = 9$, so $KN_{3,3}$ is not efficiently dominatable.

Theorem 1.2. *For $3 \leq j \leq k$, $KN_{j,k}$ is efficiently dominatable if and only if $j = 3$ and $k = 4$.*

Proof. Assume that $3 \leq j \leq k$ and $KN_{j,k}$ is efficiently dominated. Consider the lower left corner vertex, (1,1). Take S to be an efficient dominating set.

Case 1. Assume that (1,1) is an element of S. Consider (2,1), the vertex immediately to the right of (1,1). In the closed neighborhood of vertex (2,1), (1,3) and (4,2) can not be in S because each of these two vertices has a common neighbor with (1,1). So, to dominate (2,1) either (2,1) is an element of S or (3,3) is an element of S. If (2,1) is in S, (1,2) cannot be dominated by an element of S. If (3,3) is in S, to dominate (2,2), only (3,4) or (4,3) can be used since S is a packing. Without loss of generality, we can assume that (3,4) is in S. But then, (3,1) cannot be dominated by the packing S.

Case 2. Assume that (1,1) is not an element of S. Without loss of generality, we can assume that (1,1) is dominated by (2,3). Consider vertex (1,2). (1,2) cannot be in any packing S that contains (2,3), so to dominate (1,2), either (3,3) must be in S or (2,4) must be in S.

Assume that (3,3) is in S. If $k \geq 5$, (1,3) cannot be dominated by S. If $k = 4$, (2,4) cannot be dominated since every vertex in its closed neighborhood has a common neighbor with either (3,3) or (2,3).

Assume that (2,4) is an element of S. Consider the vertex (2,1). If $j \geq 4$, (2,1) cannot be dominated. If $j = 3$, then (2,1) can be dominated only by itself. So (2,1) is an element of S and $j = 3$. To dominate (2,2), (2,2) must also be in S. If $k \geq 5$, (2,5) cannot be dominated. Thus $k = 4$ and $S = \{(2,1),(2,2),(2,3),(2,4)\}$. Note that S dominates $KN_{3,4}$ and is a packing. Thus, $KN_{j,k}$ is efficiently dominatable if and only if $j = 3$ and $k = 4$. $\square$

2. Infinite Knights Graphs

In this section, we consider infinite boards. Let S and T be subsets of $\mathbb{Z}$, then $KN_{S,T}$ is the knights board defined by S cross T. For the following discussion, let $S_j = \{1, 2, \ldots, j\}$. Thus, KN_{S_j,S_k} is equivalent to the finite knights graph $KN_{j,k}$. Other special cases include $KN_{\mathbb{Z},\mathbb{Z}}$, the infinite plane which for convenience we will denote as $KN_{\infty,\infty}$, $KN_{\mathbb{Z},S_j}$, or simply $KN_{\infty,j}$ to denote the two-way infinite strip, $KN_{\mathbb{N},S_j}$ to denote the one-way infinite strip opening to the right, $KN_{\mathbb{N},\mathbb{N}}$ to denote the quarter plane and $KN_{\mathbb{Z},\mathbb{N}}$ to denote the half-plane opening upward. See [10] for a formal definition of "percentage parameters" for infinite graphs. Informally, $F\%(G)$ is simply the maximum possible percent of vertices dominatable by a packing. For example, $F\%(KN_{\infty,3}) = \frac{5}{6}$ as depicted in Figure 3 and discussed in [8].

	X	X	X	X	X	X	X		N	N		X	X	X	X	X	X	X	
		N	N	N	N		X	X			X	X		N	N	N	N		
	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	

Figure 3: One pattern that achieves $F\%(KN_{\infty,3}) = \frac{5}{6}$.

In Figure 4, we show a set of knight positions (knight locations indicated by an N) that efficiently dominates 90% of the vertices in $KN_{\infty,\infty}$, that is, $F\%(KN_{\infty,\infty}) \geq \frac{9}{10}$. In fact, as the following theorem demonstrates, equality holds. The pattern of knight positions involves using every other square in every fifth diagonal. Squares marked with an X (or an N) are those that are dominated.

X	X	X		X	X	X	X	N
X	X	N	X	X	X	X		X
X		X	X	X	X	N	X	X
N	X	X	X	X		X	X	X
X	X	X	X	N	X	X	X	X
X	X	X		X	X	X	X	N
X	X	N	X	X	X	X		X
X		X	X	X	X	N	X	X
N	X	X	X	X		X	X	X

Figure 4: $F\%(KN_{\infty,\infty}) = \frac{9}{10}$. Note that the blocked region consists of one repetition of the pattern.

Theorem 2.1. $F\%(KN_{\infty,\infty}) = \frac{9}{10}$.

Proof. Every element of an $F\%(KN_{\infty,\infty})$-set S for $KN_{\infty,\infty}$ will dominate itself plus its eight neighbors. We will show that for each element v of S, there is a vertex v' uniquely associated with v such that v' is undominated by S. For example, in Figure 4 each knight can be associated with its northeast coordinate neighbor. That is, a knight located at (i,j) is associated with $(i+1, j+1)$. It follows that $F\%(KN_{\infty,\infty}) \leq \frac{9}{10}$.

Consider an element v of S. Call this element N_1 and, without loss of generality, define its position as (0,0).

Case 1. Assume there is an $N_2 \in S$ within coordinate distance one of N_1, so that $N_2 \in \{(1,0),(0,1),(-1,0),(0,-1)\}$. Since these four positions are symmetric with respect to N_1, we can, without loss of generality, assume N_2 is at (1,0). If (0,1) is undominated by S, let $v' = (0,1)$. Otherwise, every element in $N[(0,1)]$ except (-2,2) dominates a vertex also dominated by (0,0) or (1,0), so to dominate (0,1), (-2,2) must be in the dominating set S. Similarly, to dominate (0,-1), (-2,-2) must be in the dominating set. But, both of these points cannot be in a packing because (-3,0) is a common neighbor, so either (0,1) or (0,-1) remains undominated. Similarly, to dominate (1,1), (3,2) must be in the dominating set and to dominate (1,-1), (3,-2) must be in the dominating set. Again, only one of these can be in a packing, so either (1,1) or (1,-1) remains undominated. Thus we can associate an undominated vertex from the pair (0,1) and (0,-1) with N_1 and the undominated vertex from the pair (1,1) and (1,-1) with N_2.

Case 2. Assume no vertex within coordinate distance one of N_1 is in S and $N_2 \in S$ is within coordinate distance two of N_1. Note that the only possible vertices available for N_2 are in the set $\{(-2,2),(2,2),(-2,-2),(2,-2)\}$. Since these four positions are symmetric with respect to N_1, we can, without loss of generality, assume N_2 =(-2,2). The only vertex in the closed neighborhood of (1,0) that can be in a packing with N_1 and N_2 is (2,-2). We can assume $N_3 = (2,-2)$ or else (1,0) can be associated with N_1.

To dominate (-1,-1), either $N_4 = (-3,-2)$ or $N_4 = (-2,-3)$ must be in set S. Otherwise, (-1,-1) can be associated with N_1. Similarly, to dominate (1,1), either $N_5 = (2,3)$ or $N_5 = (3,2)$ must be in set S. Otherwise, (1,1) can be associated with N_1.

Case 2A. Assume N_4 and N_5 are on opposite sides of the line $y = x$. That is, N_4 is at vertex (-2,-3) and N_5 is at vertex (2,3) or N_4 is at vertex (-3,2) and N_5 is at (3,2). Without loss of generality, assume $N_4 = (-2,-3)$ and $N_5 = (2,3)$ are in S.

If (1,-1) is not dominated by S, then N_1 can be associated with (1,-1). If (1,-1) is dominated by S, (3,-2) must be in set S. Call this vertex N_6 and call (-3,2) N_7 where $N_7 \in S$ and dominates (-1, 1). Using Case 1, N_3 can be associated with (2,-3) and N_6 can be associated with (3,-1). Similarly, N_2 can be associated with (-2,3) and N_7 can be associated with (-3,1). Also, (2,2), (1,3), (-1,-3), and (-2,-2) cannot be dominated if S is a packing. So, N_5 can be associated with (1,3), N_4 can be associated with (-1,-3), and N_1 can be associated with either (2,2) or (-2,-2).

Now consider S such that (-3,2) is not in S irrespective of whether (3,-2) is an element of S. Use N_6 =(0,5) to dominate (-1,3) or else N_2 can be associated with (-1,3) and N_1 can be associated with (-1,1). Then, (-4,-1) is an element of S which dominates (-2,0) or (-1,1) can be associated with N_2 and (-2,0) can be associated with N_1. Call (-4,-1) N_7. N_8 =(-4,4) is in S or else N_1 can be associated with (-1,1) and N_2 can be associated with (-2,3) and (-3,2). If $N_9 = (-5,4)$ is in S, then by Case 1, N_8 and N_9 have unique associations and N_1 can be associated with (-1,1). Otherwise, N_9 =(-2,7) dominates (-3,5), N_{10} =(-6,6) dominates (-4,5) and N_{11} =(6,1) dominates (-5,3). The $\{N_9, N_{10}, N_{11}\}$ pattern repeats, or as above, a Case 1 termination results in unique assignments allowing $v = (0,0)$ to be uniquely associated with $v' = (-1,1)$. Alternatively, this chain continues diagonally so that N_1 is associated with (-1,1), N_2 is associated with (-3,3), and so on.

Case 2B. Assume that N_4 and N_5 are on the same side of the line $y = x$. That is, N_4 is at vertex (-2,-3) and N_5 is at vertex (3,2) or N_4 is at vertex (-3,-2) and N_5 is at vertex (2,3). Without loss of generality, $N_4 = (-2,-3)$ and $N_5 = (3,2)$. (4,-4) must be in S to dominate (3,-2) or else N_1 can be associated with (1,-1) and N_3 can be associated with (3,-2). Call (4,-4) N_7. Then, N_8 =(5,0) must be in S to dominate (3,-1) or else N_1 can be associated with (1,-1) and N_3 can be associated with (3,-1). Similarly, N_9 =(0,-5) is in S or N_1 can be associated with (1,-1) and N_3 can be associated with (1,-3). The $\{N_3, N_8, N_9\}$ pattern repeats at $\{N_7, N_{10}, N_{11}\}$ where N_{10} =(7,-2) and N_{11} =(2,-1). Otherwise a Case 1 termination occurs resulting in unique assignments. Otherwise, this chain continues diagonally so that N_1 can be associated with (1,-1), N_3 can be associated with (3,-2), and so on.

In general, each knight in the chain is associated with the vertex diagonally up or down depending on the direction of the chain itself. These chains continue indefinitely unless two knights are placed coordinately adjacent resulting, effectively, in a termination. Associations are made from this termination up (or down) the chain as needed. Thus, for any element v of S, a unique assignment to an undominated vertex v' can be made. Therefore, $F\%(KN_{\infty,\infty}) = \frac{9}{10}$. □

Using similar assignments, we can bound the efficient domination number of all two-way infinite strip knights graphs $KN_{\infty,j}$, provided $j \geq 3$, indicating that these graphs are not efficiently dominatable. However, as shown in [8], the two-way infinite strips $KN_{\infty,1}$ and $KN_{\infty,2}$ are efficiently dominatable. That is, $F\%(KN_{\infty,1}) = F\%(KN_{\infty,2}) = 1$.

Theorem 2.2. *$F\%(KN_{\infty,j}) \leq \frac{9}{10}$ for $j \geq 3$.*

Proof. As in the proof of Theorem 2.1, each knight's location not within the three rows or columns along an edge can be associated with an undominated square. Hence, it suffices to show that each knight v within three rows or columns of the boundaries can also be uniquely associated with one undominated vertex v'. Let S denote an $F\%$-set.

Consider knights' locations within three rows or columns of a boundary. Without loss of generality, we can consider only the top three rows and for convenience we will denote them row one, row two, and row three where row one is on the 'top' boundary, row two is the next row down, and row three is the third row from the top. If $j = 3$, $F\%(KN_{\infty,j}) = \frac{5}{6}$ as shown in [8]. Assume then that $j \geq 4$.

Notice that at most two knights can be coordinately adjacent in a single row when $j \geq 4$. With this in mind, we will first consider a single knight with no coordinately adjacent neighbors and then two coordinately adjacent knights in three separate cases: the knights are in row one, in row two, and finally, in row three.

Case 1. The knights to be considered are in row one. Assume there is a knight on the boundary such that the knight has no coordinately adjacent neighbors as in Figure 5a. At most one of the a's can be in a packing, so at most one vertex coordinately adjacent to N in row one is dominated. Associate N with a horizontally adjacent undominated vertex.

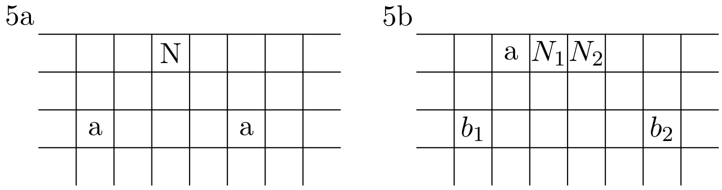

Figure 5: Case 1. The knights are in row one.

Assume there are two coordinately adjacent knights in row one as in Figure 5b. Assume b_1 is not in S. Then a and the vertex immediately below N_1 are undominated. Associate a with N_1 and the other vertex with N_2. By symmetry, if b_2 is not in S, the vertex immediately to the right of N_2 and the vertex immediately below N_2 are undominated. Associate N_2 with the vertex to the right and N_1 with the vertex below N_2. Now assume that both b_1 and b_2 are in S. Then the vertex two to the left of N_1 and the vertex two to the right of N_2 are undominated. Associate N_1 with the vertex to its left and N_2 with the vertex to its right.

Case 2. The knights to be considered are in row two. Assume there is a knight in the second row with no horizontal coordinately adjacent neighbors. Either the vertex immediately above this knight is in S or it is undominated. If it is in S, the vertices immediately to its right and left are undominated. Associate the original knight in row two with the vertex to the right and the other to the left. Otherwise, associate the knight in row two with the vertex immediately above it.

Assume there are two coordinately adjacent knights in row two. The vertices immediately above the knights are undominated. Associate each knight with the vertex above it.

Case 3. The knights to be considered are in row three. Assume there is a knight, N, in row three with no horizontal coordinately adjacent neighbors as in Figure 6a. The vertex two above N in row one cannot be dominated.

Let A be the vertex immediately above N. If A is in S, the vertex above A is associated with A by case 2, and N can be associated with either vertex immediately to the left or right of A.

Assume now that A is not in S. From case 1, a b_2 can be associated with the vertex two above N in row one under certain circumstances. Assume that neither b_2 is in S. N can now be uniquely associated with the vertex two above it. Furthermore, if at most one b_i (either b_1 or b_2) to the left side of Figure 6a and at most one b_i to the right side of Figure 6a are in S then, by case 1, each b_i in S is associated with one of its horizontal coordinate neighbors, and N can still be associated with the vertex two above it in row one. Also note that at most one b_2 is in S since the b_2's have A as a common neighbor.

It remains to define unique assignments when one b_1, b_2 pair is in S. Without loss of generality, assume it is the pair on the left side of Figure 6a. Note that by case 1, b_2 can be associated with the square two above N. If $j \leq 5$, a is undominated. Associate a with N. If $j > 5$, consider c_1 and c_2. If both c_1 and c_2 are not in S, a is undominated and N can be associated with a.

If c_1 is in S, consider the vertex, w immediately below N. Note that w cannot be in S because it has a common neighbor with b_1 which is, by assumption, an element of S. So, either w is undominated or it is dominated by d. If it is dominated, consider e. Either e is undominated or it is dominated by f. If f is not in S, associate N with either w or e, whichever is undominated. It is easy to see that w will not have an association from Theorem 2.1 with an interior knight. Being interior vertices, c_1 and d have associations defined in Theorem 2.1. Otherwise, f is an element of S and the vertex diagonally up and to the right of N is undominated. Associate N with this vertex. As interior vertices f, d, and c_1 have associations as defined in Theorem 2.1.

If c_2 is in S, w cannot be dominated. N can be associated with w and c_2 can be associated as in Theorem 2.1.

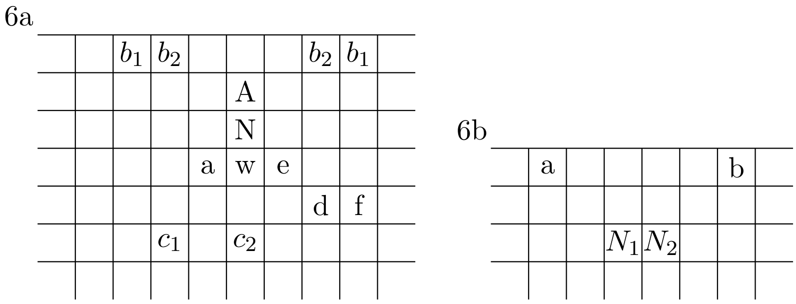

Figure 6: Case 3. The knights are in row 3.

Assume there are two horizontal coordinately adjacent knights in row three as in Figure 6b. If a is not in S, associate N_1 with the undominated vertex immediately above it. If a is in S, associate N_1 with the undominated vertex immediately below it. Similarly, N_2 can be associated with either the vertex immediately below or above depending on whether b is or is not an element of S, respectively.

Note that the above assignments are unique, and so every knight within three rows, or columns, of a boundary can be uniquely associated with an undominated vertex. Thus, $F\%(KN_{\infty,j}) \leq \frac{9}{10}$. □

Using the previous results, it is easy to see that the same bound holds for one-way infinite strips.

Corollary 2.3. $F\%(KN_{\mathbb{N},j}) \leq \frac{9}{10}$ *for* $j \geq 3$.

Proof. As in the previous results, each knight's location not within three rows or columns of an edge can be associated with an undominated vertex. Also, each knight's location within three rows or columns of an edge can be associated with an undominated vertex when the knight is not in a corner location. Thus, it suffices to show that a knight in a corner can be uniquely associated with an undominated square.

Without loss of generality, assume the knight is located in the lower left corner. Denote this vertex (1,1) and call it N_1 .

Consider (2,1) and assume it is undominated. If (2,2) is an element of S , by Theorem 2.2 (2,2) can be associated with (2,1). If (1,2) is not an element of S , it cannot be dominated and thus N_1 can be associated with it. If (1,2) is an element of S , consider (4,2). Either (4,2) is undominated or it is dominated by (6,3). If (4,2) is not dominated, N_1 can be associated with (4,2). Otherwise, (4,2) is dominated by (6,3) and (5,3) cannot be dominated. N_1 can now be associated with (5,3). Notice that from Theorem 2.2, no other knight was associated with (5,3).

Now suppose that (2,1) is dominated.

First, assume $(2,1)$ is an element of S . Call this vertex N_2 . If (2,2) is an element of S and $j \geq 4$, unique assignments can be made as is the above case. If $j = 3$, consider (5,1) which can only be dominated by (7,2). If (7,2) is not an element of S , N_1 can be associated with (5,1). Otherwise, (6,2) cannot be dominated and N_1 can be associated with (6,2). From Theorem 2.2, (6,2) could possibly have been associated with (7,3). However, (7,3) and (2,1) have a common neighbor. Furthermore, the association of N_1 with (5,1) is also unique since all other possible associations from Theorem 2.2 with (5,1) share a common neighbor with (2,1).

Now suppose that (2,2) is not an element of S . Note that (3,1) can only be dominated if $(4,3)$ is in S . Otherwise, N_2 can be associated with (3,1) and N_1 can be associated with $(1,2)$. So, assume $(4,3)$ is an element of S and call this vertex N_3 . Clearly, (1,2) and (4,1) cannot be dominated. N_1 can be associated with (1,2) and N_2 can be associated with (4,1). Notice that the only other knight location that could possibly be associated with (4,1) would be a knight located at (6,1). However, (6,1) has the common neighbor (4,2) with N_2 . So the association is unique. This scenario is depicted in Figure 7.

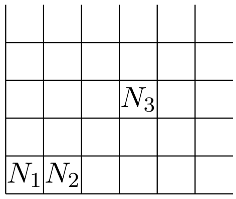

Figure 7: Case where $(1, 2) \in S$

Second, assume that (2,1) is not an element of S and is dominated. Then, (3,3) is an element of S. Call this vertex N_2 as in Figure 8. Either (3,4) is in S or else N_1 can be associated with (1,3) and N_2 was previously associated by Theorem 2.2 with (3,4) or (3,1) depending on whether (4,3) is an element of S or not, respectively. Notice that (1,5) has a common neighbor with N_1 and thus cannot be an element of S. This guarantees that the association of N with (1,3) is unique. Now assume that $(3, 4)$ is an element of S. Call it N_3. Then, N_1 can be associated with (3,1). N_2 and N_3 have unique associations by Theorem 2.2. Also by Theorem 2.2, if $(5, 1)$ is an element of S, it could also be associated with (3,1). However, (5,1) and N_1 share (3,2) as a common neighbor, thus the association of N_1 with (3,1) is unique.

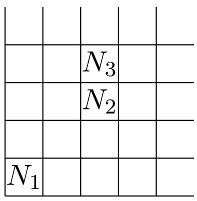

Figure 8: Case where $(3, 3) \in S$

As before, the unique association of every element of S with an undominated vertex guarantees that $F\%(KN_{\mathbb{N},j}) \leq \frac{9}{10}$. □

Clearly, this result can be extended to include half planes and quarter planes as well.

Corollary 2.4. $F\%(KN_{\mathbb{N},\mathbb{N}}) \leq \frac{9}{10})$.

Corollary 2.5. $F\%(KN_{\mathbb{Z},\mathbb{N}}) \leq \frac{9}{10}$.

3. Finite Knights Graphs

As mentioned in the introduction, $KN_{1,k}$ and $KN_{2,k}$ are efficiently dominatable. Very little is known about $KN_{3,k}$. However, if $k = 8t$, then $F(KN_{3,8t}) = 20t$. This value is achieved by relying on the patterns that are used for the two-way infinite strip when

$KN_{j,k}$	$KN_{k,1}$	$KN_{k,2}$	$KN_{8t,3}$	$KN_{3,3}$	$KN_{3,4}$	$KN_{4,4}$	$KN_{5,5}$
$F(KN_{j,k})$	k	$2k$	$20t$	7	12	12	19

Table 1: Known values of finite knights graphs.

$j = 3$. Otherwise, there are few known values of F for finite knights graphs. Table 1 summarizes these.

Beyond this, only general upper and lower bounds are currently established. Drawing upon the bound for infinite graphs, it is shown in the following theorem that the same $\frac{9}{10}$ arises in finite cases. In fact, $F(KN_{j,k}) \leq \frac{9}{10}jk$, provided the graph is sufficiently large.

Theorem 3.1. *For* $4 \leq j \leq k$, $F(KN_{j,k}) \leq \frac{9}{10}jk$

Proof. Assume there exist j and k such that $F(KN_{j,k}) > \frac{9}{10}jk$. That is, there are more than $\frac{1}{10}jk$ knights in the F-set and there are less than $\frac{1}{10}jk$ undominated vertices in the graph. So there is at least one element of the F-set such that there is no undominated vertex with which it can be associated. Call this element of the F-set N. If N is in a row or column within distance three of an edge, it can be uniquely associated with an undominated vertex as shown in the proofs of Theorem 2.2 and Corollary 5. So N must be in the interior portion of the graph. However, any such interior knight N can be associated with an undominated vertex as illustrated in the proof of Theorem 2.1. Thus, no such j and k exist and $F(KN_{j,k}) \leq \frac{9}{10}jk$. □

In large knights graphs, the interior will achieve the 90% shown in Theorem 2.1. To construct a lower bound, we subtract the three rows and columns closest to the border and consider only the interior. This is stated in the following theorem without proof.

Theorem 3.2. $F(KN_{j,k}) \geq \frac{9}{10}(jk - (6j + 6k - 36))$

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