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Queen's domination using border squares and (*A*,*B*)-restricted domination

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Note

Queen's domination using border squares and (A, B)-restricted domination

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Abstract

In this paper we introduce a variant on the long studied, highly entertaining, and very difficult problem of determining the domination number of the queen's chessboard graph, that is, determining how few queens are needed to protect all of the squares of a k by k chessboard. Motivated by the problem of minimum redundance domination, we consider the problem of determining how few queens restricted to squares on the border can be used to protect the entire chessboard. We give exact values of "border-queens" required for the k by k chessboard when $1 \leqslant k \leqslant 13$. For the general case, we present a lower bound of $k(2 - 9/2k - \sqrt{8k^2 - 49k + 49}/2k)$ and an upper bound of $k - 2$. For $k = 3t + 1$ we improve the upper bound to $2t + 1$ if $3t + 1$ is odd and $2t$ if $3t + 1$ is even.

We generalize this problem to (A, B)-restricted parameters for vertex subsets A and B of $V(G)$ where, for example, one must use only vertices in A to dominate all of B. Defining upper and lower parameters for independence, domination, and irredundance, we present a generalization of the "domination chain" of inequalities relating these parameters.
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1. Introduction

Corresponding to the chess pieces queen, rook, bishop, knight, and king there are graphs $Q_{j,k}$, $R_{j,k}$, $B_{j,k}$, $KN_{j,k}$, and $KI_{j,k}$, each of order $n = jk$, where the vertex set corresponds to the jk squares of a j by k board, and two vertices are adjacent if and only if the given chess piece can go from one of the two vertices' corresponding squares to the other corresponding square in one move. We label the vertices according to the Cartesian system and let $v_{1,1}$ be the vertex corresponding to the lower left square. Thus $v_{i,j}$ is the ith square to the right and the jth square up in the usual sense. For example, in $Q_{8,8}$ the vertex $v_{3,2}$ has the closed neighborhood $N[v_{3,2}] = \{v_{1,2}, v_{2,2}, v_{3,2}, v_{4,2}, \ldots, v_{8,2}, v_{3,1}, v_{3,3}, \ldots, v_{3,8}, v_{2,1}, v_{4,3}, v_{5,4}, \ldots, v_{8,7}, v_{4,1}, v_{2,3}, v_{1,4}\}$ with cardinality $|N[v_{3,2}]| = 1 + \deg(v_{3,2}) = 24$.

Dating back to 1848 in Bezzel [4], the literature contains hundreds of papers related to problems of the following two types. (1) What is the maximum number of mutually nonattacking chess pieces of a certain type that can be placed on the j by k board? (2) How few pieces of a given type can be used to cover the j by k board? That is, we seek (1) the independence number, $\beta(G)$, and (2) the domination number, $\gamma(G)$, of the corresponding chessboard graph G. An

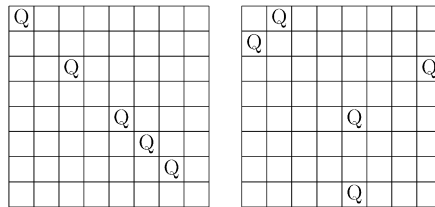*E-mail address:* philliac@email.uah.edu (A. Sinko).

Fig. 1. The left shows a γ-set having influence 126 and the right shows a γ-set having influence 116.

excellent survey article is that of Hedetniemi et al. [13]. Another wonderful exposition concerning these parameters is given by Watkins in [21]. Some more recent work on the queens domination problem has been done by Burger, Mynhardt, Weakley and others in [6–8,16,18,23,24].

Recently, influence parameters have been considered for chessboard graphs in Sinko and Slater [19,20] and Marples et al. [17]. A vertex v is considered to dominate itself and each of its adjacent vertices in its open neighborhood $N(v)$. If v has degree $\deg(v) = |N(v)|$ then v dominates $1 + \deg(v)$ vertices. For any vertex set S in a graph G, as defined in Grinstead and Slater [12], the *influence* of S is $I(S) = \sum_{v \in S} (1 + \deg(v))$, that is, the total amount of domination done by S. With the goal of dominating every vertex exactly once, one seeks a "perfect code" as defined by Biggs in [5] or "efficient dominating set" as defined by Bange et al. in [2,3]. Not every graph has such an efficient dominating set, and the efficient domination number of a graph G, as introduced in [1,3,12] and denoted $F(G)$, equals the maximum number of vertices that can be dominated by a vertex set S that does not dominate any vertex more than once. Note that S does not dominate any vertex more than once if any two vertices in S are at distance at least three, that is, S is a packing. Thus, $F(G) = \max\{I(S) : S \text{ is a packing}\}$. However, when every vertex must be dominated at least once, the redundance of graph G is defined in [12] to be the minimum influence of a dominating set. That is, the redundance of graph G is $R(G) = \min\{I(S) : S \text{ dominates } V(G)\}$. The parameters F and R and some related influence parameters are studied for chessboard graphs in [17,19].

Note, for example, that the two dominating sets for the queens chessboard $Q_{8,8}$ illustrated in Fig. 1 have influence $22 + 26 + 28 + 26 + 24 = 126$ and $22 + 22 + 22 + 28 + 22 = 116$. In fact, $R(Q_{8,8}) = 116$.

Observe that for vertex $v_{h,i}$ in $V(Q_{k,k})$ if $h \in \{1, k\}$ or $i \in \{1, k\}$ then $v_{h,i}$ is a border vertex with degree $3k - 3$. In general, for $1 \leqslant h, i \leqslant k$ let $\mathrm{ring}(v_{h,i}) = \min\{h - 1, k - h, i - 1, k - i\}$, and, for $0 \leqslant r \leqslant \lfloor k - 1/2 \rfloor$, let the rth ring of $Q_{k,k}$ be $R_r(Q_{k,k}) = \{v_{h,i} : \mathrm{ring}(v_{h,i}) = r\}$. All vertices in $R_r(Q_{k,k})$ have the same degree, namely $3k - 3 + 2r$. To minimize redundance $R(Q_{k,k})$, it seems that one should use vertices of small degree. Although one can certainly not always achieve $R(Q_{k,k})$ using only border squares, we were led to consider as a separate problem the determination of how few border vertices one can use to dominate all of $V(Q_{k,k})$. In the next section, we introduce this queens-border problem in which one seeks the minimum number of border vertices, that is, $R_0(Q_{k,k})$ vertices, that will dominate $Q_{k,k}$. (One can certainly consider the more general problem for $Q_{j,k}$.)

2. The queens-border problem

In addition to the basic problem of determining the domination number $\gamma(Q_{k,k})$, two variants have previously been studied. Cockayne and Hedetniemi [10] and Cockayne et al. [9], respectively, consider the parameters $\mathrm{diag}(Q_{k,k})$ and $\mathrm{col}(Q_{k,k})$ which are the minimum numbers of queens positions on a diagonal or column, respectively, that dominate $Q_{k,k}$. Another problem of this type is the queens-border problem that we consider here.

In particular, we restrict placement of the queens to squares in $R_0(Q_{k,k})$, the border of $Q_{k,k}$. The border-domination number of $Q_{k,k}$ is $\mathrm{bor}(Q_{k,k}) = \min\{|A| : A \text{ dominates } Q_{k,k} \text{ and } A \subseteq R_0(Q_{k,k})\}$. For example, in $Q_{3,3}$ we have $N[v_{2,2}] = V(Q_{3,3})$ so $\gamma(Q_{3,3}) = 1$. However, if $v_{i,j}$ is a border vertex then $\deg(v_{i,j}) = 6$ and $|N[v_{i,j}]| = 7$. We have $\mathrm{bor}(Q_{3,3}) = 2$, and example border dominating sets are $\{v_{1,1}, v_{2,3}\}$, $\{v_{1,1}, v_{3,3}\}$, and $\{v_{2,1}, v_{2,3}\}$. Clearly, in general, we have $\gamma(Q_{k,k}) \leqslant \mathrm{bor}(Q_{k,k})$. Also, $\gamma(Q_{4,4}) = \mathrm{bor}(Q_{4,4}) = 2$ and $\{v_{1,2}, v_{4,2}\}$ is a $\mathrm{bor}(Q_{4,4})$-set. The first several values for $\gamma(Q_{k,k})$ and $\mathrm{bor}(Q_{k,k})$ are presented in Fig. 2, along with solutions for $4 \leqslant k \leqslant 10$.

From Proposition 1 below we have $\mathrm{bor}(Q_{14,14}) \leqslant 12$ and $\mathrm{bor}(Q_{15,15}) \leqslant 13$. For domination we know that $\gamma(Q_{15,15}) \leqslant 9$. From Theorem 2 below, we get $\mathrm{bor}(Q_{14,14}) \geqslant 8$ and $\mathrm{bor}(Q_{15,15}) \geqslant 8$. Using computer searches, we have $11 \leqslant \mathrm{bor}(Q_{14,14}) \leqslant 12$ and $9 \leqslant \mathrm{bor}(Q_{15,15}) \leqslant 13$.

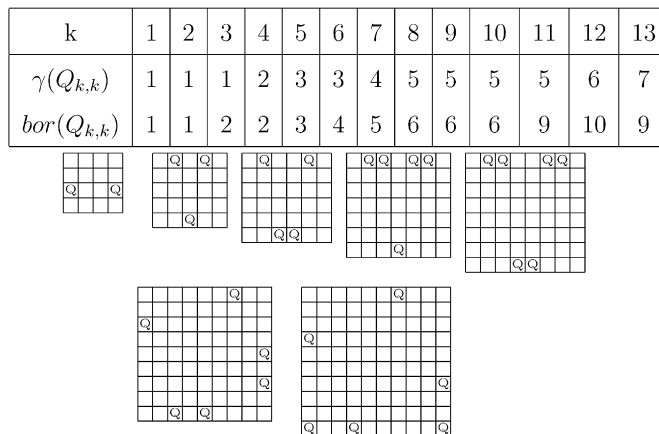| k	1	2	3	4	5	6	7	8	9	10	11	12	13
$\gamma(Q_{k,k})$	1	1	1	2	3	3	4	5	5	5	5	6	7
$bor(Q_{k,k})$	1	1	2	2	3	4	5	6	6	6	9	10	9

Fig. 2. The table gives $\gamma(Q_{k,k})$ and $bor(Q_{k,k})$ for $1 \leqslant k \leqslant 13$. Some of the possible examples are shown for $bor(Q_{k,k})$ for $4 \leqslant k \leqslant 10$.

The solutions presented in Fig. 2 for $5 \leqslant k \leqslant 8$ generalize to the next result.

Proposition 1. *For $k \geqslant 4$ we have* $bor(Q_{k,k}) \leqslant k - 2$.

Proof. For $k = 4$, let $A = \{v_{1,2}, v_{4,2}\}$. For $k = 2t + 1 \geqslant 5$, let $A = \{v_{2,k}, v_{3,k}, \ldots, v_{t,k}, v_{t+1,1}, v_{t+2,k}, \ldots, v_{2t,k}\}$. For $k = 2t \geqslant 6$, let $A = \{v_{2,k}, v_{3,k}, \ldots, v_{t-1,k}, v_{t,1}, v_{t+1,1}, v_{t+2,k}, \ldots, v_{2t-1,k}\}$. Each such A is a border-dominating set, so $bor(Q_{k,k}) \leqslant |A| = k - 2$. $\square$

Note that, while not true for $k = 9$ and 10, for $k = 11$ and 12 we again have $bor(Q_{k,k}) = k - 2$. Also, somewhat amazingly, the sequence of values is not monotone—we have $bor(Q_{13,13}) < bor(Q_{12,12})$.

A theorem of P.H. Spencer appearing in Weakley [22] states that $\gamma(Q_{k,k}) \geqslant (k - 1)/2$. Hence we have $bor(Q_{k,k}) \geqslant \gamma(Q_{k,k}) \geqslant (k - 1)/2$. The following results improve the lower bound for $bor(Q_{k,k})$.

We defined the border of $Q_{k,k}$ to be the set of vertices in $R_0(Q_{k,k}) = \{v_{h,i} : \{h, i\} \cap \{1, k\} \neq \emptyset\}$ where $1 \leqslant h, i \leqslant k$. The remaining vertices we call *interior vertices*, those within the square with corners $v_{2,2}$, $v_{2,k-1}$, $v_{k-1,k-1}$, and $v_{k-1,2}$. Note that there are $(k - 2)^2$ interior vertices of $Q_{k,k}$. Each of the corner vertices $v_{1,1}$, $v_{1,k}$, $v_{k,k}$, and $v_{k,1}$ dominates exactly $k - 2$ interior vertices, while each of the remaining border vertices dominates exactly $2k - 5$ interior vertices. It follows that $bor(Q_{k,k}) \geqslant (k - 2)^2/(2k - 5)$. By considering the redundant domination of the interior vertices (that is, the number of times interior vertices get dominated more than once), we improve the lower bound for $bor(Q_{k,k})$ to essentially $(2 - \sqrt{2})k \approx .585786k$, as follows.

Theorem 2. $bor(Q_{k,k}) \geqslant a_k \cdot k$ *where* $a_k = 2 - 9/2k - \sqrt{8k^2 - 40k + 49}/2k$. *(Note that* $\lim_{k \to \infty} a_k = 2 - \sqrt{2}$.)

Proof. We will show that the number of border queens required to dominate the $(k - 2)^2$ interior vertices is at least $a_k \cdot k$. Assume some set A of border locations dominates all $(k - 2)^2$ interior vertices of $Q_{k,k}$ and that $|A| = j$. Define $A_c = \{v_{hi} \in A : h \in \{1, k\}\}$ for $1 \leqslant i \leqslant k$ and $A_r = \{v_{hi} \in A : i \in \{1, k\}\}$ for $1 \leqslant h \leqslant k$. Using Proposition 1, we can assume that $1 \leqslant j \leqslant k - 2$. Initially, suppose that $A \cap \{v_{1,1}, v_{1,k}, v_{k,k}, v_{k,1}\} = \emptyset$. By rotating the board 90 degrees, if necessary, we can assume that $|A_c| \geqslant |A_r|$. That is, the combined number t of vertices in A_c of $Q_{k,k}$ satisfies $t \geqslant j/2$. Finally, the result can be verified for $k \leqslant 4$, so we assume $k \geqslant 5$.

As noted, each $v \in A$ dominates $2k - 5$ interior vertices. We consider the number of times two elements of A dominate a common square. Let v_1 and v_2 be two of the t elements of A_c in A. If they are on the same row, say $v_1 = v_{1,i}$ and $v_2 = v_{k,i}$, then they both dominate the $(k - 2) \geqslant 3$ interior vertices of row i. If v_1 and v_2 are on different rows, they each dominate one square in the other's row. The redundant domination of interior vertices by these t elements of A is at least $2 \cdot \binom{t}{2}$. For the $j - t$ elements of A_r in A we count the redundant domination of these with the t elements of A_c also in A. For each of the t elements of A_c in A, if the other end of its row is not in A we get $j - t$ common interior vertices dominated. Consider the redundant domination if the other end is also in A. There are at most $t/2$ rows with

both ends in A which implies that there are $t/2(k-4)$ remaining redundancies from the horizontal pairs and $t/2(j-t)$ redundancies vertically. But, $t/2(k-4) - t/2(j-t) \geqslant j/4(j-2) - j/4(j/2)$ since $t \geqslant j/2$ and $k \geqslant (j+2)$. This value is positive if $j \geqslant 2$ which implies that $k \geqslant 4$. Since $k \geqslant 5$, the total redundance is at least $2\binom{t}{2} + t(j-t)$. It follows that:

$$j(2k-5) - \left[2\binom{t}{2} + (j-t)t\right] \geqslant (k-2)^2. \tag{1}$$

From (1), letting $j = a_k k$, and noting that the left-hand side is maximized when $t = j/2$ we get

$$k^2 a_k^2 + (9k - 4k^2)a_k + (2k^2 - 8k + 8) \leqslant 0. \tag{2}$$

Hence, we get $a_k = 2 - 9/2k - \sqrt{8k^2 - 40k + 49}/2k$.

Clearly, a minimal set of border vertices dominating the $(k-2)^2$ interior vertices of $Q_{k,k}$ does not contain both $v_{1,1}$ and $v_{k,k}$ or both $v_{1,k}$ and $v_{k,1}$.

If $|A \cap \{v_{1,1}, v_{1,k}, v_{k,k}, v_{k,1}\}| = 1$, then we can assume $v_{k,k} \in A$ and there are $t \geqslant j - 1/2$ other A_c vertices in A. As above, we get

$$(j-1)(2k-5) + (k-2) - \left[2\binom{t}{2} + (j-t)t\right] \geqslant (k-2)^2. \tag{3}$$

From (3), letting $a_k k = j$, the left-hand side is maximized at $t = (j-1)/2$, and we get

$$k^2 a_k^2 + (8k - 4k^2)a_k + (2k^2 - 6k + 9) \leqslant 0. \tag{4}$$

Hence, $a_k = 2 - 4/k - \sqrt{8k^2 - 40k - 28}/2k \geqslant a_k$.

If $|A \cap \{v_{1,1}, v_{1,k}, v_{k,k}, v_{k,1}\}| = 2$, then we can assume $\{v_{k,1}, v_{k,k}\} \subseteq A$ and there are $t \geqslant (j-2)/2$ other A_c vertices in A. As above, we get

$$(j-2)(2k-5) + 2(k-2) - \left[2\binom{t}{2} + (j-t)t\right] \geqslant (k-2)^2. \tag{5}$$

From (5), letting $a_k k = j$, the left-hand side is maximized at $t = (j-2)/2$, and we get

$$k^2 a_k^2 + (7k - 4k^2)a_k + (2k^2 - 4k - 2) \leqslant 0. \tag{6}$$

Hence, $a_k = 2 - 7/2k - \sqrt{8k^2 - 40k + 57}/2k \geqslant a_k$. Thus, $j \geqslant a_k = 2 - 9/2k - \sqrt{8k^2 - 40k + 49}/2k$, and the proof is complete. $\square$

Note that from Theorem 2 we get $a_{10} = .490519$, so we can only conclude $\mathrm{bor}(Q_{10,10}) \geqslant 5$. A slightly more detailed analysis using the same techniques shows $\mathrm{bor}(Q_{10,10}) \geqslant 6$.

Proposition 3. $\mathrm{bor}(Q_{10,10}) = 6$.

Proof. As shown in Fig. 1, $\mathrm{bor}(Q_{10,10}) \leqslant 6$. By the above statements, $\mathrm{bor}(Q_{10,10}) \geqslant 5$. To show $\mathrm{bor}(Q_{10,10}) \geqslant 6$ we will consider similar techniques used in Theorem 2. Here, we have $k = 10$ and each noncorner border vertex dominates $2 \cdot 10 - 5 = 15$ interior vertices.

Take $A \subseteq V(Q_{10,10})$ to be a dominating set with $j = 5$ border queens.

First, assume $|A \cap \{v_{1,1}, v_{1,10}, v_{10,1}, v_{10,10}\}| = \emptyset$. Let j, the total number of vertices in the dominating set A, be 5. Recall from above that t, the number of vertices in A on the left and right edges, satisfies $t \geqslant j/2$. Then, $j(2 \cdot 10 - 5) - [2\binom{t}{2} + (j-t)t] \geqslant (10-2)^2$. The left side of this equation is maximized when $t = j/2$, so let $t = 3$. Then the equation reduces to $63 \geqslant 64$, a contradiction.

Next, assume $|A \cap \{v_{1,1}, v_{1,10}, v_{10,1}, v_{10,10}\}| = 1$. The equation becomes $(j-1)(2 \cdot 10 - 5) + (10-2) - [2\binom{t}{2} + (j-1-t)t] \geqslant (10-2)^2$ by Eq. (3) above. Here $t \geqslant j - 1/2$, and the left-hand side is maximized when $t = (j-1)/2 = 2$. Then the equation reduces to $62 \geqslant 64$, another contradiction.
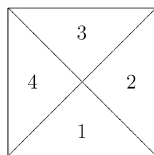
Fig. 3. The $Q_{3t+1,3t+1}$ graph divided into four disjoint sections.

Last, assume $|A \cap \{v_{1,1}, v_{1,10}, v_{10,1}, v_{10,10}\}| = 2$. Then, $(j-2)(2 \cdot 10 - 5) + 2(10 - 2) = 61$ which is clearly less than $(10 - 2)^2$ even without considering redundance.

So, $\text{bor}(Q_{10,10}) \geqslant 6$. Therefore, $\text{bor}(Q_{10,10}) = 6$. □

Recall that the only stated upper bound for $\text{bor}(Q_{k,k})$ is $k - 2$. However, for the infinite family of $Q_{3t+1,3t+1}$, an improved upper bound can be given.

Theorem 4. $\text{bor}(Q_{3t+1,3t+1}) \leqslant 2t + 1$ *if* $3t + 1$ *is odd and* $\text{bor}(Q_{3t+1,3t+1}) \leqslant 2t$ *if* $3t + 1$ *is even.*

Proof. To form a border dominating set $A \subseteq V(Q_{3t+1,3t+1})$, first let $\{v_{1,1}, v_{3t+1,1}\} \subseteq A$, that is, the two bottom corner vertices are elements of A. Then, add $\{v_{1,4}, v_{3t-2,1}, v_{3t+1,3t-2}, v_{4,3t+1}\}$ to A. Note that these are the fourth vertices on each side moving clockwise. Next, the seventh vertex on each side is placed in A, that is, $\{v_{1,7}, v_{3t-5,1}, v_{3t+1,3t-5}, v_{7,3t+1}\}$. This pattern continues adding every third set of vertices. If $3t + 1$ is odd, the last set to be added contains $v_{1,3t+2/2}$. If $3t + 1$ is even, the last set to be added contains $v_{1,\lfloor\frac{3t}{2}\rfloor}$. Also, if $3t + 1$ is odd, only the first three vertices of this last set in the pattern are placed in A, and if $3t + 1$ is even, the usual four are added.

We will show that A dominates $V(Q_{3t+1,3t+1})$. Assume A does not dominate $Q_{3t+1,3t+1}$. Then there exists vertex $v_{i,j}$ such that $N[v_{i,j}] \cap A = \emptyset$. Consider the set $R = \{4, 7, \ldots, 3t - 5, 3t - 2\}$. Notice that, except for the two corner vertices, this set represents the nonborder label of the vertices in A. Since $v_{i,j}$ is not dominated, $i \neq j$ and $j \neq 3t + 2 - i$ as these are the vertices dominated by the two corner vertices in A. Also, $i, j \neq 3c + 1$ where $c = 1, \ldots, t - 1$ as these vertices are dominated horizontally or vertically, respectively, by vertices in A.

Consider $Q_{3t+1,3t+1}$ divided into four sections as shown in Fig. 3.

Consider Section 1, that is $2 < i < j < 3t + 2 - i$. The border neighbors of $v_{i,j}$ are $v_{i+j-1,1}$, $v_{i-j+3t+1,3t+1}$, $v_{1,j+i-1}$, and $v_{1,j-i+1}$ beginning with the upper left neighbor and moving clockwise.

Case 1: $i = 3a, j = 3b$ or $i = 3a + 2, j = 3b + 1$ where $a \neq b$. Then, the lower left neighbor is $v_{1,3(b-a)+1}$. Since $b > a, 3(b - a) + 1 > 2$. So, $3(b - a) + 1$ is an element of R. Also, since $2 < i < j < 3t + 2 - i$, $v_{1,3(b-a)+1} \in A$.

Case 2: $i = 3a, j = 3b + 2$. Then the lower right neighbor is $v_{1,3(b+a)+1}$ and the upper left neighbor is $v_{3(b+a)+1,1}$. Note that, at most, $b < t - 1$ when $a = 1$ which implies that $3(b + a) + 1 < 3t + 1 \Rightarrow 3(b - a) + 1 \leqslant 3t - 2$. So, $3(b + a) + 1$ is an element of R. With the symmetry on the bottom row and first column from selecting vertices for A, $\{v_{3(b+a)+1,1}, v_{1,3(b+1)+1}\} \cap A \neq \emptyset$.

Case 3: $i = 3a + 2, j = 3b$. Then, the lower right neighbor is $v_{1,3(b+a)+1}$ and the upper left neighbor is $v_{3(b+a)+1,1}$. As above, the restrictions on i and j imply that $3(b + a) + 1 \leqslant 3t - 2$. Thus $3(b + a) + 1$ is an element of R and, from symmetry, $\{v_{1,3(b+a)+1}, v_{3(b+a)+1,1}\} \cap A \neq \emptyset$.

Similarly, the same can be shown for the other three sections. So $N[v_{i,j}] \cap A \neq \emptyset$. Thus, vertex $v_{i,j}$ is dominated, and A dominates $Q_{3t+1,3t+1}$.

Therefore, there exists a dominating set A such that $\text{bor}(Q_{3t+1,3t+1}) \leqslant 2t + 1$ if $3t + 1$ is odd and $\text{bor}(Q_{3t+1,3t+1}) \leqslant 2t$ if $3t + 1$ is even. □

3. Generalization and other parameters

For the border-domination of $Q_{k,k}$ introduced in this paper, the problem is to dominate all of the vertices while using only vertices from a specified set A, in this case $A = R_0(Q_{k,k})$. More generally, we define the following problem.

(A, B)-*restricted domination*: Given vertex subsets A and B of $V(G)$ the (A, B)-domination number is the minimum cardinality of a subset $A' \subseteq A$ that dominates B. We let $\gamma_{(A,B)}(G) = \min\{|A'| : A' \subseteq A, B \subseteq N[A']\}$. If $B = V(G)$, we let $\gamma_A(G) = \gamma_{(A,V(G))}(G)$. If A does not dominate B, then $\gamma_{(A,B)}(G) = \infty$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$BOR(Q_{k,k}) \geq$	1	1	2	4	4	5	6	8	9	11	12	14	16	17	19

16	17	18	19	20
20	22	24	25	27

Fig. 4. Some lower bounds for $BOR(Q_{k,k})$.

An example of (A, B)-restricted domination would be bipartite domination as discussed by Hedetniemi and Laskar in [14,15]. For our problem, $\mathrm{bor}(Q_{k,k}) = \gamma_{R_0(Q_{k,k})}(Q_{k,k})$.

Likewise, other graphical parameters can become A-restricted or (A, B)-restricted. While A-restricted independence might not seem to directly offer questions of interest (see below), A-restricted packing does. We can let $\rho_A(G)$ be the maximum cardinality of a subset A' of A that is a packing in G. For the upper-domination parameter Γ we define $\Gamma_{(A,B)}(G)$ to be the maximum cardinality of a subset A' of A such that A' is a minimal dominating set for B. Some lower bounds, mostly from computer generated examples, for $\Gamma_{B_0(Q_{k,k})} \equiv BOR(Q_{k,k})$ are presented in Fig. 4.

Also of interest are the as yet uninvestigated lower and upper border-irredundance parameters $\mathrm{ir}_{R_0}(Q_{k,k})$ and $\mathrm{IR}_{R_0}(Q_{k,k})$. Generalizing again, for subsets A and B the upper and lower (A, B)-irredundance numbers, denoted $\mathrm{IR}_{(A,B)}(G)$ and $\mathrm{ir}_{(A,B)}(G)$, are the maximum and minimum cardinalities, respectively, of a subset $A' \subseteq B$ which is maximal with respect to the property that for each $v \in A'$ there is a vertex $w \in B$ with $N[w] \cap A' = \{v\}$. That is, each v in A' is the sole dominator of a vertex w in B, in which case w is called a private neighbor of v. If $A \cap N[B] = \emptyset$, then $\mathrm{ir}_{(A,B)}(G) = \mathrm{IR}_{(A,B)}(G) = 0$. The next result follows directly.

Theorem 5. *For any graph G and any subsets A and B, we have* $\mathrm{ir}_{(A,B)}(G) \leqslant \gamma_{(A,B)}(G) \leqslant \Gamma_{(A,B)}(G)$, *and if* $B \subseteq N[A]$ *(that is, $\Gamma_{(A,B)}(G)$ is finite) then $\Gamma_{(A,B)}(G) \leqslant IR_{(A,B)}(G)$.*

Considering the concept of independence, the upper and lower independence numbers $\beta(G)$ and $i(G)$ are the maximum and minimum cardinalities, respectively, of maximal independent sets. Recall that an independent set is maximal if and only if it minimally dominates. We can therefore generalize parameters β and i as follows. Let A and B be vertex subsets of $V(G)$, then the upper and lower (A, B)-independence parameters $\beta_{(A,B)}(G)$ and $i_{(A,B)}(G)$ are the maximum and minimum cardinalities of an independent set $A' \subseteq A$ such that A' minimally dominates B. Our concluding result generalizes the domination chain in Cockayne et al. [11].

Theorem 6. *If A and B are subsets of $V(G)$ and A contains an independent subset A' that dominates B, then* $\mathrm{ir}_{(A,B)}(G) \leqslant \gamma_{(A,B)}(G) \leqslant i_{(A,B)}(G) \leqslant \beta_{(A,B)}(G) \leqslant \Gamma_{(A,B)}(G) \leqslant \mathrm{IR}_{(A,B)}(G)$.

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