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Homomorphisms for equidistance relations

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HOMOMOPHISMS FOR EQUIDISTANCE RELATIONS

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Abstract. This paper presents necessary and sufficient conditions for the existence of homomorphisms for equidistance relations in terms of the closed subsystems (the Fundamental Theorem of Homomorphisms). Further it shows that every closed subsystem of a 1-point homogeneous equidistance system is a coset of a unique homomorphism. Affine spaces and other incidence geometries can be seen as examples of equidistance systems.

The notion of a flat, as in Dembrowski (1968), provides a natural definition of subgeometry generalizing lines and planes. This paper starts from a different geometric perspective, equidistance, which is taken as a primitive. Sibley (1977) proves that to every equi distance relation E there corresponds a metric d so that aa'Ebb' iff d(a,a')= d(b,b'), given a suitable cardinality restriction. See Sibley (1979) and (1981) as well for further properties of equidistance relations. See Gratzer (1968: 80-82,98,224) for a discussion of notions of subsystems and homomorphisms.

Definitions.

An equidistance relation E on a non-empty set A is a subset of ordered 4-tuples of A, written abEcd, so that i) abEba, ii) abEcc iff a = b, and iii) E is an eqivalence relation on AxA, the ordered pairs of A.

Note that each such relation E corresponds to a partition of the complete graph on A into classes of different distances. An E-system (A,E) is an equidistance relation E on a non-empty set A. Although we can form a subsystem by restricting E to any subset of A, this provides no sense of closure, unlike the following definition. Given (A,E) and $B \subset A$, $B \neq \emptyset$; (B,E) is a sub-E-system of (A,E) iff for all a, b, $c \in B$ and $d \in A$, if abEcd; then $d \in B$.

Given two E-systems (A,E) and (A',E'), a function $h:A \to A'$ is a homomorphism from (A,E) into (A',E') iff for all a, b, c, $d \in A$, if abEcd; then $h(a)h(b)E'h(c)h(d)$. If h is one-to-one, then it is an embedding. If h is onto and both h and h^{-1} are embeddings, then (A,E) and (A',E') are isomorphic. If E and E' are equidistance relations on A, E is weaker than E' iff for all a, b, c, $d \in A$, abEcd implies abE'cd. Equivalently, $E \subset E'$, when they are considered as subsets of A^4. Clearly, for $E \subset E'$ with $E \neq E'$ on A, the identity is an embedding from (A,E) onto (A,E'), even though (A,E) and (A,E') are not isomorphic. This mildly irritating situation occurs quite generally for homomorphisms on relational systems and requires an appropriate qualification in Theorem 2, below. For a homomorphism h from (A,E) into (A',E'), and for $a \in A$, define $H_a = \{b \in A: h(a) = h(b)\}$ to be a coset of h. Just as in algebra, without regularity conditions, the cosets can be haphazard. 1-point homogeneous E-systems provide a particularly nice setting for homomorphisms.

A bijection s on A is an isometry of (A,E) iff for all $a,b \in A$, abEs(a)s(b). (A,E) is 1-point homogeneous iff for all $a, b \in A$, there is an isometry s so that $s(a) = b$. (A,E) is regular iff every point of a has the same number of points at a given "distance" from it. A bijection s on A is an automorphism or similiarity of (A,E) iff for all a, b, c, $d \in A$, if abEcd; then s(a)s(b)Es(c)s(d). Given (A,E), two subsets of A, $\{b_i: i \in I\}$ and $\{c_i: i \in I\}$, are congruent iff for all i, $j \in I$, $b_i b_j E c_i c_j$. Given (A,E), two subsets of A, $\{b_i: i \in I\}$ and $\{c_i; i \in I\}$, are isometric iff there is an isometry s of (A,E) so that for all $i \in I$, $s(b_i) = c_i$. Note that congruent is weaker than isometric.

Properties of Homomorphisms.

Prop. 1 Every coset of a homomorphism is a sub-E-system. Further, if abEcd and $H_a = H_b$, then $H_c = H_d$.

Proof. If abEcd, then h(a)h(b)Eh(c)h(d). If $h(a) = h(b)$, property ii) of an equidistance relation forces $h(c) = h(d)$. From a, b, $c \in H_a$ and abEcd, the above gives. Thus every coset is a sub-E-system ∎

Property 1 above provides necessary conditions for a homomorphism. The construction below shows that these conditions are also sufficient, which is the content of Theorem 2, the Fundamental Theorem of Homomorphisms. Suppose (A,E) is an E-system with disjoint sub-E-systems $\{(H_a,E): a \in A\}$ which partition A,E). The extra condition in Prop. 1 is that the distances inside any of the sub-E-systems do not appear between points of different sub-E-systems. More precisely, a partition of (A,E) by sub-E-systems $\{(H_a,E): a\in A\}$ is <u>normal</u> iff for all $a, b, c, d \in A$, if $abEcd$ and $H_a = H_b$, then $H_c = H_d$.

Given a normal partition, we construct the "factor system" as follows. Let $A/H = \{H_a: a\in A\}$ and define E^* on A/H by $H_aH_bE^*H_cH_d$ iff there are $x' \in H_x$ so that $a'b'Ec'd'$. E^* is not necessarily an equidistance relation. Let E/H be the transitive closure of E^*. Then it is easy to see that E/H is an equidistance relation: i) for all $a, b\in A$ $abEba$, so H_aH_bE/HH_bH_a. ii) By definition of E^*, $H_aH_aE^*H_bH_b$ and so for E/H as well. The condition of a normal partition guarantees that $H_aH_bE^*H_cH_c$ implies $H_a = H_b$. Thus the same holds for E/H. iii) E^* is reflective and symmetric, so E/H is. E/H is tansitive.

Hence $(A/H,E/H)$ is an E-system. The canonical mapping $h(a) = H_a$ is obviously a homomorphism from (A,E) onto $(A/H,E/H)$. The rest of the Fundamental Theorem states how these canonical homomorphisms describe all others. Suppose $h': A\rightarrow A'$ is a homomorphism from (A,E) into (A',E') with the same sub-E-systems $\{(H_a,E): a\in A\}$ as cosets. The mapping $k: A/H\rightarrow A'$ given by $k(H_a) = h'(a)$ is clearly well defined and an embedding from $(A/H,E/H)$ into (A',E'). In effect, E/H is the weakest equidistance on A/H which can be the homomorphic image of (A,E) using these cosets. Any stronger relation on A/H can also be such an image and, up to isomorphism, those are all images.

<u>Theorem</u> 2 (Fundamental Theorem of Homomorphisms). Let (A,E) be an E-system with a normal partition of sub-E-systems $A/H = \{(H_a,E): a\in A\}$. For E/H as defined above, $h: A\rightarrow A/H$ given by $h(a) = H_a$ is a homomorphism from (A,E) onto $(A/H,E/H)$. Further, if $h': A\rightarrow A'$ is a homomorphism from (A,E) onto (A',E') with cosets the elements of A/H, then there is E'' on A/H so that $E/H\subset E''$ and $(A/H, E'')$ and (A',E') are isomorphic.

The Isomorphism Theorems below follow readily upon noting that the non-empty intersection of sub-E-systems is again a sub-E-system.

<u>Prop</u>. 3 (First Isomorphism Theorem). Let (A,E) be an E-system, (B,E) a sub-E-system and $\{(H_a,E): a\in A\}$ a noraml partition of (A,E). Then $\{(B\cap H_a,E): a\in A\}$ ia a normal partition of (B,E) and $(B/B\cap H,E/B\cap H)$ is isomorpnic to the sub-E-system $(\{H_a: a\in B\},E/H)$ of $(A/H,E/H)$.

<u>Prop</u>. 4 (Second Isomorphic Theorem). Let (A,E) be an E-system and $\{(H_a,E): a\in A\}$ and $\{(K_a,E): a\in A\}$ be normal partitions of (A,E) so that for all $a\in A$, $H_a\subset K_a$. Then $(A/K,E/K)$ is isomorphic to $(A/H/K/H, E/H/K)$,

where E/H/K is the equidistance relation on A/H/K/H determined by the Fundamental Theorem of Homomorphism from (A/H,E/H).

The special case of 1-point homogeneous spaces deserves seperate mention. Note the similiarity of Prop. 6 below with La Grange's THeorem and the fact that for commutative groups, every subgroup is normal.

Prop. 5 If (A,E) is 1-point homogeneous and h: $A \longrightarrow A'$ is a homomorphism from (A,E) onto (A',E'), then i) all cosets are isometric (and so congruent) sub-E-systems and ii) (A',E') is also 1-point homogeneous.

Proof. Let a and b be in A and s an isometry with s(a)= b. Then $s(H_a)$ is isometric and so congruent to H_a and $b \in s(H_a)$. To show $s(H_a) = H_b$, consider the following. For all $b' \in s(H_a)$, there is $a' = s^{-1}(b') \in H_a$ so that aa'Ebb'. Hence by Prop. 1, $b' \in H_b$, that is, $s(H_a) \subset H_b$. Similiarly, we have $s^{-1}(H_b) \subset H_a$. These force $s(H_a) = H_b$, which is part i). for part ii), note first that if $E'' \subset E'$ on A' and s' is an isometry for (A',E"), then s' is an isometry for (A',E') also. Hence, by the Fundamental Theorem of Homomorphisms, it suffices to show that (A/H,E/H) is 1-point homogeneous. Let H_a, $H_b \in A/H$. For s an isometry of (A,E) with s(a) = b, let $S(H_c) = H_{s(c)}$ on A/H. Clearly, S is an isomerty and $S(H_a) = H_b$ ∎

Prop. 6 Every sub-E-system of a 1-point homogeneous E-system is the coset of a unique (canonical) homomorphism. Hence the order of a sub-E-system divides the order of the 1-point homogeneous E-system.

Proof. Let (B,E) be any sub-E-system of (A,E) and let s be any isometry of (A,E). Then, by the same reasoning as for part i) of Prop. 5, (s(B),E) is also a sub-E-system of (A,E). Further, these sub-E-systems are congruent. Hence they are either disjoint or identical. Further, {(s(B),E): s is an isometry of (A,E)} is a normal partition of (A,E). Both the uniqueness of the Homomorphism and "LaGrange's Theorem" on the order of sub-E-systems now follow immedicately ∎

Since 1-point homogeneity was used in Prop. 6 only to insure congruent cosets, we immediately have this Corollary.

Cor. 7 If an E-system can be partitioned in congruent sub-E-system, then that partition is normal.

Prop. 5, but not 6, generalizes some for regular E-systems.

Prop. 8 If (A,E) is regular and h: $A \longrightarrow A'$ is a homomorphism from (A,E) onto (A',E'), then i) all cosets are equinumerous (but not necessarily congruent). Hence the order of the sub-E-systems of a normal partition divides the order of the E-system. ii) if each coset is finite, then (A',E') is also regular.

Proof. Let H_a and H_b be any two cosets of (A,E) and $a' \in H_a$. By Prop. 1, $\{a'' : aa'Eaa''\} \subset H_a$ and by regualrity, $\{b'' : aa'Ebb''\}$ is equinumerous with it. Further, by Prop. 1, $\{b'' : aa'Ebb''\} \subset H_b$. We can do the same for each $a^* \in H_a$ to obtain an injection of H_a into H_b. By symmetry, we get part i). Part ii) depends on the fact that division of a cardinal number by another is well defined provided the second one is finite (and non-zero). In effect, the homomorphism collapses the equivalence classes of "distances" as well as the points. But the finiteness of the cosets insures that the new equivalent classes of "distances" from different cosets will still be equinumerous ∎

Examples. The examples below illustrate the propositions above and their limitations. See Sibley [5] and [6] for a description of various 1-point homogeneous E-systems.

Example 1. If (G,*) is a group, define $/a/ = \{a, a^{-1}\}$ and abEcd iff $/a*b^{-1}/ = /c*d^{-1}/$. This generalises absolute values and distances from the reals. (G,E) is 1-point homogeneous with isometries $s_a(x) = x*a$, for $a \in G$. Sibley [6] generalizes this. The sub-E-systems of (G,E) consist exactly of the right cosets of the subgroups of (G,*) because a subset is a subgroup iff it is closed under products of the form $a*b^{-1}$ which correspcnds the definition of E. Thus Prop. 6 provides the mildly suprising result that even subgroups which are not normal determine normal partitions of the corresponding sub-E-systems.

Example 2 On a set with n elements, let E be the equidistance relation with only one non-zero distances. Then there are no non-trival sub-E-systems. This E-system is clearly 1-point homogeneous since any bijection is an isometry.

Example 3 Let (B,E') be the six point regular E-system determined by Diagram 1. There are five non-zero distances represented by straight, dashed, dotted, wavy, and double lines. Any pair of points forms a sub-E-system. Three such congruent sub-E-systems, say, $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, form a normal partition illustrating Cor. 7. However, the partition $\{1,2\}$, $\{3,5\}$, $\{4,6\}$, is not normal even though these sub-E-

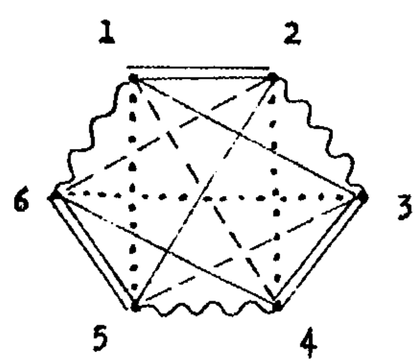

Diagram 1

systems are equinumerous. Thus the converse of Prop. 8 i) need not hold.

Example 4. Let (C,E") be the seven point regular E-system determined by determined by Diagram 2. There are three non-zero distances represented by straight, dashed, and wavy lines. The subset $\{a, b, b'\}$ forms a sub-E-system but is not part of a normal partition. Thus LaGrange's Theorem on the order of sub-E-systems need not hold for regular E-systems, although Prop. 8 i) shows it does hold for "normal" sub-E-systems. (C,E") is also useful in showing that the cosets of a regular geometry need not be congruent.

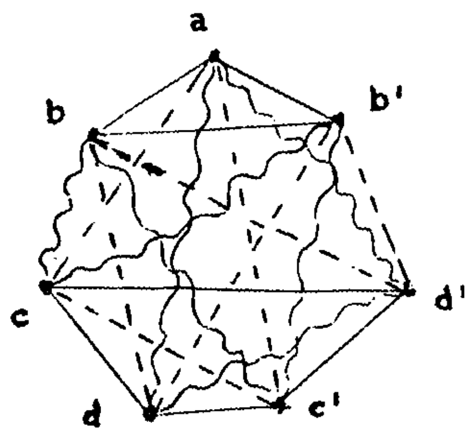

Diagram 2

Let E be the equidistance relation described in example 1 of the seven element group, C_7. Note that E also has 3 non-zero distances. Define E* on $C \cup C_7$ as follows. E* has four non-zero distances so that (C,E*) is a sub-E-system isomorphic to (C,E") and (C_7,E^*) is a sub-E-system isomorphic to (C_7,E) with the same three distances of E* used in each. The fourth distance of E* is for pairs of points where one is from C and the other from C_7. By construction, (C,E*) and (C_7,E^*) form a normal partition although they are not congruent.

Example 5. To show that the finiteness condition of Prop. 8 ii) is needed, defined E on $\{1, 2, 3, 4\} \times Z$ with three non-zero distances as follows. (See Diagram 3). "Wavy" distance for (a,i) (b,j) iff /a-b/ = 1. "Dashed" distance for (a,i) (b,j) iff /a-b/ $\geq$ 2. "Circular" distance (in diagram 3) for (a,i) (a,j) iff $i \neq j$. Then the sets $\{1\} \times Z$, $\{2\} \times Z$, $\{3\} \times Z$, and $\{4\} \times Z$ are congruent sub-E-systems. However, the equidistance relation on the four cosets, $\{H_1, H_2, H_3, H_4\}$ given by the Funuamental Theorem of Homorphisms is not regular because H_1 has one "wavy" edge and H_2 has two "wavy" edges.

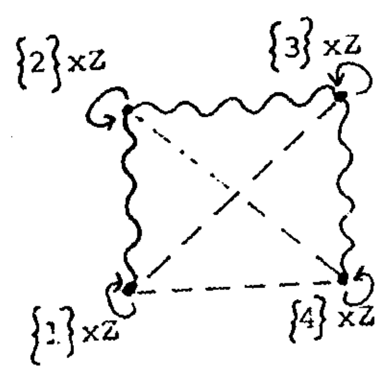

Diagram 3

Lines and Equidistance.

An unusual interpretation of equidistance encompasses many geometries based on lines (incidence geometries). I will use "geometry" to refer to a set with designated subsets called lines, while retaining "E-systems" for the same set with the corresponding equidistance relation. The most important such geometries satisfythe properties that there is a unique line incident with two distinct points and each line is incident with at least two points. For a non-empty set G with a family of lines $\mathcal{L}$ satisfying these properties, call $(G,\mathcal{L})$ unilinear. Let $\overline{ab}$ be the line incident with a and b, two distinct points. Define an equidistance relation E on G by abEcd iff either ($a = b$ and $c = d$) or ($a \neq b$, $c \neq d$, and $\overline{ab} = \overline{cd}$). A bijection of G is an affine transformation iff the image of a line is a line. The condition that every line has at least two points is needed since the equidistance relation could not distinguish between a singleton that was a line and one that wasn't. Clearly, the equidistance relation E contains exactly the same information as $\mathcal{L}$ does. Thus Prop. 9 below is immediate. Prop. 10 characterizes those equidistance relations which correspond to unilinear incidence geometries.

Prop. 9 If $(G,\mathcal{L})$ is unilinear and E is the corresponding equidistance relation, then the affine transformations of $(G,\mathcal{L})$ are exactly the automorphisms of (G,E).

Prop. 10 If E is the equidistance relation of a unilinear geometry on a non-empty set G, then for all distinct a, b, $c \in G$,

(*) if abEbc , then abEac.

Further, if for all distinct a, b, $c \in G$, (*) holds for an equidistance relation E', then E' determines a set of lines $\mathcal{L}$ by $\overline{ab} = \{c : abE'ac\} \cup \{a\}$ for $a \neq b$ so that $(G,\mathcal{L})$ is unilinear.

Proof. Let $(G,\mathcal{L})$ be unilinear and E the equidistance relation derived from $\mathcal{L}$. Then for distinct elements a, b, $c \in G$, abEbc iff there is a unique $L \in \mathcal{L}$ so that a, b, $c \in L$. This implies (*). Note that each line forms a sub-E-system with exactly one non-zero distance. For the second part, let E' be an equidistance relation on G satisfying (*) for distinct a, b, $c \in G$. For $a \neq b$, define $\overline{ab} = \{c : abE'ac\} \cup \{a\}$. Let $\mathcal{L} = \{\overline{ab} : a \neq b\}$. Clearly, each line in $\mathcal{L}$ has at least two elements and every two elements have a line incident with them. Note that $\overline{ab}$ is the sub-E'-system generated by a and b and that this sub-E'-system has exactly one non-zero distance by (*). Hence for $c \neq d$ and $c, d \in \overline{ab}$, we have $\overline{cd} = \overline{ab}$. Thus there is a unique line incident with two distinct point, proving unilinearity ∎

If we consider the equivalence classes of E, then the property (*) is very like a transitivity condition. That is, if ab and bc are in a class, then ac is in the same class. Further, the definition of an equidistance relation provides the corresponding "symmetric-like" and "reflexive like" conditions. That is, if ab is in a class, then ba is in the same class and all aa are in the same class. Note that the second part of Prop. 10 is not a converse of the first part. From E' satisfying (*), we obtain $\mathcal{L}$ which in turn gives the equidistance relation E of the first part. One can readily see that $E \subset E'$. Further, there is a certain "parallelism" among the lines of $\mathcal{L}$ which, as sub-E'-systems, have the same distance. This becomes most apparent with affine spaces. Let $(G,\mathcal{L})$ be an affine space with "//" meaning "parallel". Define E' on G by abE'cd iff either (a = b and c = d) or ($a \neq b$, $c \neq d$, and $\overline{ab}//\overline{cd}$). An affine transformation s on $(G,\mathcal{L})$ is a <u>homothety</u> iff for all $L \in \mathcal{L}$, s(L)//L. See Blumenthal (1980, pages 73-84) for an explanation of the first Desargues property for affine planes, its necessity and its uses for the proof of the following theorem.

<u>Prop</u>. 11 If $(G,\mathcal{L})$ is an affine plane with the first Desargues property and E and E' are as defined above, then i) the homotheties of $(G,\mathcal{L})$ are the isometries of (G,E'), ii) (G,E') is 1-point homogeneous and iii) (G,E') has the same automorphisms as (G,E).

<u>Proof</u>. Part i) is immediate from the definitions of the terms and E'. ii) Blumenthal (1980) constructs the usual addition on equivalence classes of vectors using the first Desargues property . This addition uses the familiar parallelogram law and forms a commutative group. Each equivalence class of vectors corresponds to a translation. That is, for all a,b, $c \in G$, there is a unique $d \in G$ such that $\vec{ab} = \vec{cd}$ and so we can define $t_{a,b}(c) = d$. By the parallelogram law these translations are homotheties. Thus (G,E') is 1-point homogeneous. Part iii) is a consequence of the fact that the relation of parallelism is determined by the set of lines in a affine plane ∎

Prop. 11 can be extended to some affine spaces, including those over fields. The existence of skew lines in higher dimensions implies that additional structure besides the first Desargues property is needed. The cosets of homomorphisms of such affine spaces are the "flats" of the geometry, that is, the right (and left) cosets of the subspaces. The translations of such affine spaces form commutative groups which can be readily identified with the points of the spaces. (Wolff [7] shows that a uniquely determined parallelogram structure is equivalent to a commutative group.) Let E" be the equidistance relation defined on the points of the affine space which is derived from the group of translations as in example 1. Then $E'' \subset E'$ because abE"cd implies $\overline{ab}//\overline{cd}$ by the parallelogram property (or $a = b$ and $c = d$) which implies abE'cd.

The equidistance relation E' of Example 3 above imitates the relation E' for affine spaces in the use of a sort of parallelism. This suggests the following generalization in which // is an undefined relation on $\mathcal{L}$. $(G,\mathcal{L},//)$ is <u>//-unilinear</u> iff i) $(G,\mathcal{L})$ is unilinear, ii) // is an equivalence relation, and iii) for all $a \in G$ and $L \in \mathcal{L}$, there is a unique $M \in \mathcal{L}$: $a \in M$ and L//M. For $(G,\mathcal{L},//)$ //-unilinear, define E' on G by abE'cd iff either ($a = b$ and $c = d$) or ($a \neq b$, $c \neq d$, and $\overline{ab}//\overline{cd}$). Note L//M and $L \neq M$ imply $L \cap M = \emptyset$.

<u>Prop</u>. 12 If $(G,\mathcal{L},//)$ is //-unilinear and E' is defined as above, then any set of parallel lines forms a normal partition of (G,E').

<u>Proof</u>. By the definition of E', each line is clearly a sub-E-system. By properties ii) and iii) of //-unilinear, we have a partition. Again, by definition of E', the partition is normal ∎

Unlike affine spaces, //-unilinear geometries can have different numbers of points on lines. The easiest such example comes from eliminating one point from an affine space (in which every line has at least three points). This of course entails that there can be non-parallel lines with no intersection, even in the same "plane".

Spherical geometries are not unilinear. However, every line which goes through a point goes through its antipode. Thus the usual identification of opposite points gives single elliptic geometries as homomorphic images, which are unilinear. All unilinear geometries can be generalized in a similiar manner, yielding generalizations of Prop. 9, 10 and 12.

Many geometries already have a metric defined on them. In particular, Euclidean and hyperbolic geometries have both the incidence structure and the metric. To simplify, we will consider only for Euclidean spaces how the unusual equidistance relation defined from the lines relates with the usual equidistance relation defined from the metric. Let E_m be the equidistance relation defined from the Euclidean metric on a space with at least two dimensions. (One dimension is trivial.) Let E_1 be the equidistance relation defined from the lines. E_m is strong enough to determine the metric. (Sibley [4] proves this in a more general context.) It is well known that the metric determines the lines. Thus every similarity (automorphism of E_m) must be a linear transformation (automorphism of E_1).

For a Euclidean space, consider also the equidistance relation E_g derived from the group of translations of the space and E_a derived from the affine structure. (See Example 1 and the definition before Prop. 11 for their definitions). Note $abE_g cd$ iff $abE_m cd$ and $abE_a cd$. That is, E_m considers the lengths, E_a considers the directions and E_g considers both, as vectors do. For $i \in \{a, g, 1, m\}$, write I_i for the isometries of E_i. Then I_1 contains only the identity, I_a contains the homotheties, I_m contains the usual isometries and $I_g = I_m \cap I_a$ contains the translations and central symmetries. Note that there are discontinuous linear transformations in the groups of automorphisms of E_a', E_g', and E_1. The usual affine transformations form the subgroup of continuous automorphisms of these.

The possible homomorphic images of a Euclidean space using these different equidistance relations reveals another aspect of their interactions. The homomorphic images of (R^k, E_g) are simply the E-systems corresponding to groups which can be homomorphic images of $(R^k, +)$. At the other extreme, a homomorphism from (R^k, E_m) for $K \geq 2$ is either an embedding or maps all of R^k to a single point. In between are the other equidistance relations E_1 and E_a. They act like vector spaces in that the homomorphic images of (R^k, E_1) and (R^k, E_a) are lower dimensional Euclidean spaces of the same form. This expresses in another way that the incidence structure of Euclidean space corresponds very closely to the algebraic structure while the usual metric ties everything together so to speak, into the analytic structure based on neighbourhoods.

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