Computing local constants for CM elliptic curves

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COMPUTING LOCAL CONSTANTS
FOR CM ELLIPTIC CURVES

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ABSTRACT. Let $E/k$ be an elliptic curve with CM by $\mathcal{O}$. We determine a formula for (a generalization of) the arithmetic local constant of $[5]$ at almost all primes of good reduction. We apply this formula to the CM curves defined over $\mathbb{Q}$ and are able to describe extensions $F/\mathbb{Q}$ over which the $\mathcal{O}$-rank of $E$ grows.

1. Introduction. Let $p$ be an odd rational prime. Let $k \subset K \subset L$ be a tower of number fields, with $K/k$ quadratic, $L/K$ $p$-power cyclic and $L/k$ Galois with a dihedral Galois group, i.e., a lift of $1 \neq c \in \text{Gal}(K/k)$ acts by conjugation on $g \in \text{Gal}(L/K)$ as $cgc^{-1} = g^{-1}$. In [5] Mazur and Rubin define arithmetic local constants $\delta_v$, one for each prime $v$ of $K$, which describe the growth in $\mathbb{Z}$-rank of $E$ over the extension $L/K$. Specifically (cf., [5, Theorem 6.4]), for $\chi : \text{Gal}(L/K) \hookrightarrow \overline{\mathbb{Q}}^{	imes}$ an injective character and $S$ a set of primes of $K$ containing all primes above $p$, all primes ramified in $L/K$ and all primes where $E$ has bad reduction,

\[(1.1) \quad \text{rank}_{\mathbb{Z}[\chi]} E(L)^{\chi} - \text{rank}_{\mathbb{Z}} E(K) \equiv \sum_{v \in S} \delta_v \pmod{2}.\]

To phrase their result this way, we must assume the Shafarevich-Tate conjecture\(^1\), and we keep this assumption throughout.

In [1], the theory of arithmetic local constants is generalized to address the $\mathcal{O}$-rank of varieties with complex multiplication (CM) by an order $\mathcal{O}$, and we continue in that direction with specific attention to the elliptic curve case. Following [1], we assume that $\mathcal{O} \subset \text{End}_K(E)$ is the maximal order in a quadratic imaginary field $K$, $p$ is unramified in $\mathcal{O}$, and $\mathcal{O}^c = \mathcal{O}^{\dagger} = \mathcal{O}$ where $^{\dagger}$ indicates the action of the Rosati

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involution (see [6, subsection I.14]). When $K \not\subset k$, these assumptions imply $K = kK$.

Our present aim is to provide a simple formula for the local constants $\delta_v$ (see Definition 2.2) for primes $v \nmid p$ of good reduction. We then will use a result [1, Section 6] which generalizes (1.1), with $\mathbb{Z}$ replaced by $\mathcal{O}$, to determine conditions under which the $\mathcal{O}$-rank of $E$ grows. In Section 3 we will describe, via class field theory, dihedral extensions $F/\mathbb{Q}$ which satisfy those conditions, in order to give some concrete setting to the results of Section 2.

2. Computing the local constant. Suppose $p$ splits in $\mathcal{O}$, i.e., $p\mathcal{O} = p_1 p_2$, with $p_1 \neq p_2$. We denote $R = \mathcal{O}/p\mathcal{O}$ and $R_i = \mathcal{O}/p_i$ for $i = 1, 2$, so that $R \cong R_1 \oplus R_2$.

Definition 2.1. If $M$ is an $\mathcal{O}$-module of exponent $p$, define the $R$-rank of $M$ by

\[
\text{rank}_R M := (\text{rank}_{R_1} M \otimes_R R_1, \text{rank}_{R_2} M \otimes_R R_2).
\]

The following definition is the same as in [1, 5]. Fix a prime $v$ of $K$, and let $u$ and $w$ be primes of $k$ below $v$ and of $L$ above $v$, respectively. Denote $k_u$, $K_v$ and $L_w$ for the completions of $k$, $K$ and $L$ at $u$, $v$ and $w$, respectively. If $L_w \neq K_v$, let $L'_w$ be the extension of $K_v$ inside $L_w$ with $[L_w : L'_w] = p$, and otherwise let $L'_w = L_w = K_v$.

Definition 2.2. Define the arithmetic local constant $\delta_v := \delta(v, E, L/K)$ by

\[
\delta_v \equiv \frac{E(K_v)}{E(K_v) \cap N_{L_w/L'_w} E(L_w)} \pmod{2}.
\]

Now, we will consider primes $v$ of $K$ such that $E$ has good reduction at $v$, $v \nmid p$, $v^c = v$ and $v$ ramifies in $L/K$ (corresponding to [5, Lemma 6.6]). Under these conditions, [5, Theorem 5.6] shows that

\[
(2.1) \quad \dim_{\mathbb{F}_p} \frac{E(K_v)}{E(K_v) \cap N_{L_w/L'_w} E(L_w)} \equiv \dim_{\mathbb{F}_p} E(K_v)[p] \pmod{2}.
\]
Proposition 2.4 below shows that we are able to replace $\dim_{\mathbb{F}_p}$ by $\text{rank}_R$ in (2.1). We first need Lemma 2.3, which follows Lemmas 5.4 and 5.5 of [5], and our proof is meant only to address the change from $\dim_{\mathbb{F}_p}$ to $\text{rank}_R$.

Let $\mathcal{K}$ and $\mathcal{L}$ be finite extensions of $\mathbb{Q}_\ell$, with $\ell \neq p$, and suppose $\mathcal{L}/\mathcal{K}$ is a finite extension.

**Lemma 2.3.** Suppose $\mathcal{L}/\mathcal{K}$ is cyclic of degree $p$, $E$ is defined over $\mathcal{K}$ and has good reduction. Then:

(i) $\text{rank}_R E(\mathcal{K})/pE(\mathcal{K}) = \text{rank}_R E(\mathcal{K})[p]$.

(ii) If $\mathcal{L}/\mathcal{K}$ is ramified, then $E(\mathcal{K})/pE(\mathcal{K}) = E(\mathcal{L})/pE(\mathcal{L})$ and $N_{\mathcal{L}/\mathcal{K}} E(\mathcal{L}) = pE(\mathcal{K})$.

(iii) If $\mathcal{L}/\mathcal{K}$ is unramified, then $N_{\mathcal{L}/\mathcal{K}} E(\mathcal{L}) = E(\mathcal{K})$.

**Proof.** When $\ell \neq p$, we have $E(\mathcal{K})/pE(\mathcal{K}) = E(\mathcal{K})[p^\infty]/pE(\mathcal{K})[p^\infty]$. Since $E(\mathcal{K})[p^\infty]$ is finite, (i) follows from the exact sequence of $\mathcal{O}$-modules

$$
0 \longrightarrow E(\mathcal{K})[p] \longrightarrow E(\mathcal{K})[p^\infty] \longrightarrow pE(\mathcal{K})[p^\infty] \longrightarrow 0.
$$

The content of (ii) and (iii) is on the level of sets, so the proof is exactly as in [5, Lemma 5.5].

We return to the notation of Definition 2.2.

**Proposition 2.4.** If $v \nmid p$ and $L_w/K_v$ is nontrivial and totally ramified, then

$$
\delta_v \equiv \text{rank}_RE(K_v)[p] \pmod{2}.
$$

**Proof.** As in [5, Proof of Theorem 5.6], Lemma 2.3 (ii) yields $N_{L_w/L'_w} E(L_w) = pE(L'_w)$, and hence $E(K_v) \cap pE(L'_w) = pE(K_v)$. So by Definition 2.2 and Lemma 2.3 (i),

$$
\delta_v \equiv \text{rank}_R \frac{E(K_v)}{pE(K_v)} \equiv \text{rank}_R E(K_v)[p] \pmod{2}.
$$
Now, fix a prime $v$ of $K$. We denote $\kappa_u$ for the residue field of $k_u$, $q = \#\kappa_u$ for the size of finite field $\kappa_u$ and $\tilde{E}$ for the reduction of $E$ to $\kappa_u$.

**Proposition 2.5.** Suppose $v \nmid p$, $v$ is ramified in $L/K$ and $v^c = v$. If $E$ has good reduction at $v$, then $\delta_v \equiv (1,1)$ if and only if $p \mid \#\tilde{E}(\kappa_u)$.

**Proof.** We follow the notation of [5, Lemma 6.6]. Since $v^c = v$, we know that $K_v/k_u$ is quadratic, and it is unramified by [5, Lemma 6.5 (ii)]. Let $\Phi$ be the Frobenius generator of $\text{Gal}(K_v^{ur}/k_u)$, so $\Phi^2$ is the Frobenius of $\text{Gal}(K_v^{ur}/K_v)$.

The proof of Lemma 6.6 [5] shows that the product of the eigenvalues $\alpha, \beta$ of $\Phi$ on $E[p]$ is $-1$. Also, $E(K_v)[p] = E[p]^{\Phi^2=1}$ is equal (as a set) to $E[p]$ or is trivial depending on whether or not $\{\alpha, \beta\} = \{1, -1\}$, respectively. Since $E$ has CM by $\mathcal{O}$, $E[p]$ is a rank 1 $R$-module (see, e.g., [10, Section II.1]), so the former case yields

$$
\delta_v \equiv \text{rank}_RE(K_v)[p] = (1,1) \pmod{2}.
$$

By assumption, $v \nmid p$, so $p$ is prime to the characteristic of $\kappa_u$, and therefore the reduction map restricted to $p$-torsion is injective [9, subsection VII.3]. We also know $E[p]$ is unramified [9, subsection VII.4], and so the eigenvalues of $\Phi$ acting on $E[p]$ or is trivial depending on whether or not $\{\alpha, \beta\} = \{1, -1\}$, respectively. Since $E$ has CM by $\mathcal{O}$, $E[p]$ is a rank 1 $R$-module (see, e.g., [10, Section II.1]), so the former case yields

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$$
\delta_v \equiv \text{rank}_RE(K_v)[p] = (1,1) \pmod{2}.
$$

**Corollary 2.6.** If $K \not\subset k$, then $\delta_v \equiv (1,1)$.

**Proof.** To see that $p \mid \#\tilde{E}(\kappa_u)$, we show that $a = 0$ under our assumptions on $v$, where $a = q + 1 - \#\tilde{E}(\kappa_u)$ as above. The theory of complex multiplication gives $a = \pi_u + \pi_u$ for some $\pi_u \in \mathcal{O}$ such that $\pi_u \pi_u = q$ (see, e.g., [3, Theorem 14.16], [10, subsection II.10] or [8] for a thorough discussion). As $K \not\subset k$, we have $K = kK$, and we let $\psi = \psi_{E/K}$ be the Grössencharacter associated to $E$ and $K$ (see
[10, subsection II.9] or [4]). By comparing their effect on $K(E[\ell])$, where $\ell$ is prime to $v$, we see that $\psi(v)^c = \psi(v^c)$, and since $v = v^c$, we have that $\psi(v)$ is fixed by $c$. It follows that $\psi(v)$ is rational, the corresponding $\pi_v \in \mathcal{O} \subset \text{End}_K(E)$ is integral, and in fact, $\pi_v = \pm q$ by degree arguments. In addition, $\pi_v^2 = \pi_v$, and we will see that $\pi_v = \sqrt{-q}$ is purely imaginary. Indeed, $\pi_v$ having no real part implies $a = \pi_u + \bar{\pi}_u = 0$; hence,

$$\#\bar{E}(\kappa_u) \equiv q + 1 \equiv 0 \pmod{p}$$

and $\delta_v \equiv (1, 1)$ by Proposition 2.5.

Suppose instead that $\pi_u = \sqrt{q}$ is real. If, in addition, we suppose $\pi_u$ is integral then the reduction $\phi_q \in \text{End}(\bar{E})$ of $\pi_u$ would commute with all endomorphisms of $\bar{E}$. As $K \not\subset k$, there is some $\rho \in \text{End}_K(E)$ such that $\rho \neq \rho^c$, and hence, $\rho \neq \rho^c$. Thus, for some $P \in \bar{E}(\kappa_u)$, $P^c = P$ and $\rho(P^c) \neq \rho^c(P)$. As the action of $c$ on $\kappa_u$ coincides with that of Frobenius $\tilde{\Phi}$, it follows that $\tilde{\rho}$ does not commute with $\tilde{\phi}$, and in turn $\tilde{\rho}$ does not commute with the Frobenius endomorphism $\phi_q \in \text{End}(\bar{E})$ induced by $\tilde{\Phi}$.

If $\pi_u = \sqrt{q}$ is real and irrational, then $k \subset \mathbb{Q}(\pi_u)k \subset K$ and so $c \in \text{Gal}(K/k)$ acts non-trivially on $\pi_u$, i.e., $\pi_u^c = -\sqrt{q}$. It follows that

$$q = N_{K/Q}(\pi_u) = \pi_u \pi_u^c = -q,$$

which is a contradiction, and we conclude $\pi_u$ is purely imaginary as desired. □

Define a set $\mathcal{S}_L$ of primes $v$ of $K$ by

$$\mathcal{S}_L := \{v \mid p, \text{or } v \text{ ramifies in } L/K, \text{ or where } E \text{ has bad reduction}\}.$$

**Theorem 2.7** [1, Theorem 6.1]. Let $\chi : \text{Gal}(L/K) \hookrightarrow \mathbb{Q}^\times$ be an injective character and $\mathcal{O}[\chi]$ the extension of $\mathcal{O}$ by the values of $\chi$. Assuming the Shafarevich-Tate conjecture,

$$\text{rank}_{\mathcal{O}[\chi]}E(L)^{\chi} - \text{rank}_{\mathcal{O}}E(K) \equiv \sum_{v \in \mathcal{S}_L} \delta_v \pmod{2}.$$
We now consider a dihedral tower \( k \subset K \subset F \) where \( F/K \) is \( p \)-power abelian. Following [5, Section 3], we note that there is a bijection between cyclic extensions \( L/K \) in \( F \) and irreducible rational representations \( \rho_L \) of \( G = \text{Gal}(F/K) \). The semi-simple group ring \( K[G] \) decomposes as
\[
K[G] \cong \bigoplus_L K[G]_L,
\]
where \( K[G]_L \) is the \( \rho_L \)-isotypic component of \( K[G] \). For each \( L \), it suffices to deal with an injective character \( \chi : \text{Gal}(L/K) \hookrightarrow \mathbb{Q}^\times \) appearing in the direct-sum decomposition of \( \rho_L \otimes \mathbb{Q}^\times \), and \( \text{rank}_{\mathcal{O}[\chi]} E(F)^\chi \) is independent \(^5\) of the choice of \( \chi \).

**Theorem 2.8.** Assume \( K \not\subset k \). Suppose that, for every prime \( v \) satisfying \( v^c = v \) and which ramifies in \( F/K \), we have \( v \nmid p \) and \( E \) has good reduction at \( v \). For \( m \) equal to the number of such \( v \), if \( \text{rank}_{\mathcal{O}} E(K) + m \) is odd, then
\[
\text{rank}_{\mathcal{O}} E(F) \geq [F : K].
\]

**Proof.** Fix a cyclic extension \( L/K \) inside \( F \). If \( v \) is a prime of \( K \) and \( v^c \neq v \), then \( \delta_v \equiv \delta_{v^c} \) and hence \( \delta_v + \delta_{v^c} \equiv (0, 0) \pmod{2} \) by [5, Lemma 5.1]. If \( v^c = v \) and \( v \) is unramified in \( L/K \), then \( v \) splits completely in \( L/K \) by [5, Lemma 6.5 (i)]. It follows that \( N_{L/v}/L_{v} \) is surjective, and so \( \delta_v \equiv (0, 0) \) by Definition 2.2. The remaining primes \( v \) are precisely those named in the assumption, so Proposition 2.6 gives \( \sum_v \delta_v \equiv (m, m) \pmod{2} \). Thus,
\[
\text{rank}_{\mathcal{O}[\chi]} E(L)^\chi \equiv \text{rank}_{\mathcal{O}} E(K) + m \pmod{2},
\]
and we have assumed that the right-hand side is odd.

From [5, Corollary 3.7], it follows that
\[
\text{rank}_{\mathcal{O}} E(F) = \sum_L \left( \dim_{\mathbb{Q}} \rho_L \right) \cdot \left( \text{rank}_{\mathcal{O}[\chi]} E(L)^\chi \right).
\]
As the previous paragraph applies for every cyclic \( L/K \) in \( F \) we see from the decomposition of \( K[G] \) that \( E(F) \otimes \mathbb{Q} \) contains a submodule isomorphic to \( K[G] \), and the claim follows. \( \square \)
3. CM elliptic curves defined over \( \mathbb{Q} \). Here, we will consider the CM elliptic curves \( E \) defined over \( \mathbb{Q} \) (as in [10, A.3]). For each \( E \), our aim is to determine \( 7 \) examples of dihedral towers \( \mathbb{Q} \subset K \subset F \) over which, according to Theorem 2.8, the \( \mathcal{O} \)-rank of \( E \) grows. As we have assumed \( \mathcal{O} \subset \text{End}_K(E) \), we will consider towers in which \( K = K \) (see Section 1). All of our calculations will be done using Sage [11].

Let \( E_D/\mathbb{Q} \) be the elliptic curve of minimal conductor \( 8 \) defined over \( \mathbb{Q} \) with CM by \( K_D = \mathbb{Q}(\sqrt{-D}) \). We determine computationally \( 9 \) rank \( \mathbb{Z} E_D(K_D) \), and for \( D = 3 \), we see that this group is finite. For \( D = 4 \), the situation is less certain, as Sage only tells us that \( E_D(\mathbb{Q}) \) is finite and \( \text{rank}_{\mathbb{Z}} E_D(K_D) \leq 2 \). For each of the remaining CM curves \( E_D \) defined over \( \mathbb{Q} \), one can (provably) calculate that \( \text{rank}_{\mathbb{Z}} E_D(\mathbb{Q}) = 1 \). We also have that \( \text{rank}_{\mathbb{Z}} E_D(K_D) \geq \text{rank}_{\mathbb{Z}} E_D(\mathbb{Q}) = 1 \) and \( \text{rank}_{\mathbb{Z}} E_D(K_D) \) cannot be even, so \( \text{rank}_{\mathbb{Z}} E_D(K_D) = 1 \). For \( D = 8, 11, 19, 43, 67 \) and 163, Sage gives an upper bound \( 7 \) of 3 for \( \text{rank}_{\mathbb{Z}} E_D(K_D) \) and so, for these \( D \), we can conclude that in fact \( \text{rank}_{\mathcal{O}} E_D(K_D) = 1 \).

3.1. Dihedral extensions of \( \mathbb{Q} \). Recall that \( p \) is a fixed odd rational prime. Presently, we also fix \( D \in \{3, 4, 7, \ldots, 163\} \), and let \( E = E_D, K = K_D \). We are interested in abelian extensions \( F/K \) which are dihedral over \( \mathbb{Q} \), and these are exactly the extensions contained in the ring class fields of \( K \) (see [3, Theorem 9.18]).

Let \( \mathcal{O}_f \) be an order in \( \mathcal{O}_K \) of conductor \( f \). We have a simple formula for the class number \( h(\mathcal{O}_f) \) of \( \mathcal{O}_f \) using, for example, [3, Theorem 7.24], and noting that, we have \( h(\mathcal{O}_K) = 1 \),

\[
h(\mathcal{O}_f) = \frac{f}{[\mathcal{O}_K^\times : \mathcal{O}_f^\times]} \cdot \prod_{\text{primes } \ell | f} \left(1 - \left(\frac{-D}{\ell}\right) \frac{1}{\ell}\right).
\]

For \( D \neq 3, 4 \), we have \( \mathcal{O}_K^\times = \{\pm 1\} \) and, for \( D = 4 \), we have \#\( \mathcal{O}_K^\times = 4 \), so in both of these cases \( [\mathcal{O}_K^\times : \mathcal{O}_f^\times] \) is prime to \( p \). For \( D = 3 \), one can show that \( [\mathcal{O}_K^\times : \mathcal{O}_f^\times] = 3 \) when \( f > 1 \). The following paragraphs require only minor adjustments for the case \( p = D = 3 \).

Taking \( f \) to be an odd rational prime such that \((-D/f) = \pm 1\), the class number becomes \( h(\mathcal{O}_f) = f \mp 1 \), and so the ring class field \( H_{\mathcal{O}_f} \) associated to \( \mathcal{O}_f \) is an abelian extension of \( K \) of degree \( f \mp 1 \). Thus, for \( f \equiv \pm 1 \mod p \), we have a (non-trivial) \( p \)-power subextension \( F/K \) which is dihedral over \( \mathbb{Q} \).
Next, we need to understand the ramification in $F/K$. As $K$ has class number 1, we know there are no unramified extensions of $K$, and so we must ensure that $F$ satisfies the hypotheses of Theorem 2.8. A prime $v$ of $K$ ramifies in $H_{O_f}/K$ if and only if $v \mid fO_K$ (see, for example, [3, Exercise 9.20] and recall $f$ is odd). If we choose $f$ so that $-D$ is not a square (mod $f$), $f$ is inert in $K/Q$, and so $fO_K$ is prime and, moreover, the only prime that ramifies in $H_{O_f}/K$. If $fO_K$ does not ramify in $F/K$, then the $p$-extension $F/K$ is contained in the Hilbert class field $H_K$ of $K$. As $H_K = K$, this is impossible, so $fO_K$ ramifies in $F/K$ and no other primes ramify in $F/K$. Taking $f$ such that $f \not| D$ and $-D$ is a square (mod $f$), we have that $f$ is not inert and does not ramify in $K/Q$. As in the previous case, the primes of $K$ above $f$ both ramify in the $p$-extension $F/K$ contained in $H_{O_f}$.

Now, suppose rank$_{O_E}(K)$ is odd$^{10}$. To apply Theorem 2.8, we must have an even number $m$ of primes $v$ such that $v^c = v$, $v$ ramifies in $F/K$, $E$ has good reduction at $v$ and for which $p \mid \#\tilde{E}(\mathbb{Z}/f\mathbb{Z})$. First, we can guarantee $m = 0$ if the only primes $v$ which ramify in $F/K$ do not satisfy $v^c = v$, e.g., taking $f \not| D$ with $(-D/f) = 1$. Table 3.1 gives, for each $D$ and for $p = 3, 5, 7$, the smallest prime $f$ which gives an extension of degree $p$ following this recipe. We note that we do not need Proposition 2.5 for this case.

If we wish to allow for primes $v$ satisfying $v^c = v$, we choose two $p$-extensions $F_1$ and $F_2$ from two distinct rational primes $f_i$ as above with $f_i \equiv -1$ (mod $p$) and $(-D/f_i) = -1$, for $i = 1, 2$. The compositum $F = F_1F_2$ will satisfy our requirements. Indeed, firstly $F$ is an abelian $p$-extension of $K$ and is contained in the ring class field $H_{O_{f_1,f_2}}$, hence dihedral over $Q$ with only $f_1O_K$ and $f_2O_K$ ramifying in $F/K$. Secondly, as each $f_i$ is inert in $K/Q$, each is a supersingular prime for $E$ (this follows from the arguments in Corollary 2.6) and hence $p$ divides $\#\tilde{E}(\mathbb{Z}/f_i\mathbb{Z}) = f_i + 1$. Thus, $E$ and the $p$-extension $F/K$ satisfy the hypotheses of Theorem 2.8. Table 3.2 below gives, for each $D$ and for $p = 3, 5, 7$, the smallest pair of distinct primes $f_1, f_2$ which give extensions of degree $p^2$ following this recipe.

Next, suppose rank$_{O_E}(K)$ is even.$^{11}$ In this case, we need $m$ to be odd in order to apply Theorem 2.8. The same ideas as above still work, and in Table 3.3 we list, for each $D$ and for $p = 3, 5, 7$, the smallest prime $f$ for which Theorem 2.8 guarantees rank $\geq p$. 

Remark 3.1. Though there are algorithms in the literature to compute the defining polynomial of a class field (e.g., [2, Section 6], [3, subsection 11-3]) and such computational problems are of interest independently, we make no attempt here to explicitly determine the ring class fields $H_{O_f}$. As is apparent from Table 3.2, our method of determining a field to which Theorem 2.8 applies involves ring class fields of large degree in a computationally inefficient way.

**TABLE 3.1. Case $m = 0$.**

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**TABLE 3.2. Case $m = 2$.**

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<th>$f_1$</th>
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<th>$[F : K]$</th>
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</table>
Acknowledgments. The first author would like to thank Colorado College for support during his Riley-scholar post-doctoral fellowship. We would also like to thank the referee for the comments and suggestions, particularly regarding Proposition 2.5 and Corollary 2.6.

ENDNOTES

1. Without this assumption, all statements regarding $\mathcal{O}$-rank of $E$ would be replaced by analogous statements regarding $\mathcal{O} \otimes \mathbb{Z}_p$-corank of the $p^\infty$-Selmer group $\text{Sel}_p^\infty(E/K)$ of $E$.

2. The simpler case of $p$ being inert in $\mathcal{O}$, i.e., $\mathcal{O}/p\mathcal{O}$ is a field, is treated similarly.

3. That $a = 0$ in this case is known (see [10, Exercise 2.30], [4, Section 4, Theorem 10] or [7, Theorem 7.46] for generalization to higher dimensional abelian varieties); we include an argument for completeness.

4. The case $\pi_u = -\sqrt{q}$ follows the same argument.

5. We could instead write that $\dim_{\mathbb{Q}}(E(F) \otimes \mathbb{Q})^\times$ is independent of the choice of $\chi$.

6. The case $K \subset k$ is similar, with $m$ equal to the number of $v$ such that $p | \#\tilde{E}(\kappa_u)$.

7. Determined up to the correspondence of class field theory.

8. See [10, page 483], with $f = 1$ (in Silverman’s notation), for a Weierstrauss equation.

9. Specifically with Sage’s interface to John Cremona’s ‘mwraky’ and Denis Simon’s ‘simon_two_descent’.
10. The cases $D = 8, 11, \ldots, 163$, and possibly $D = 4, 7$.

11. The case $D = 3$, and possibly $D = 4, 7$.

REFERENCES


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