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Rhombic Penrose tilings can be 3-colored

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NOTES

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The four-color theorem tells us that any tiling in the plane can be 4-colored, where adjacent tiles are to get different colors. Anyone who tries to color a Penrose tiling will wonder whether they are all 3-colorable; this question was first proposed by John H. Conway [**2**, p. 27]. We show, by a very simple argument, that any tiling by Penrose rhombs is 3-colorable; our method yields an algorithm, the results of which are illustrated in Figure 1. We were led to this result by a *Mathematica* implementation of a 4-coloring algorithm for plane maps based on Kempe chains; see [**5**] and [**7**]. It had no difficulty in 3-coloring rhombus tilings. For background on Penrose tilings see [**2**], [**4**], and [**8**]. Note that there is no simple characterization of the 3-colorable planar maps, and the problem of recognizing 3-colorable planar maps is NP-complete [**3**].

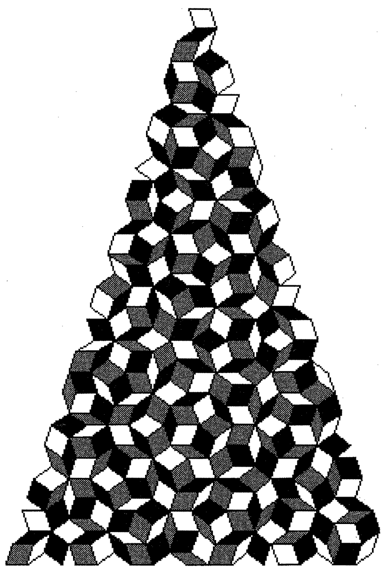

Figure 1. A 3-coloring of a Penrose rhomb tiling with 228 tiles.

In fact, our main result is much more general, and has nothing to do with Penrose geometry. Call a map with polygonal countries *tidy* if whenever two countries meet, they do so either in a single point, or in a complete edge (or several complete edges) of each polygon of the pair.

Theorem. *Any tidy plane map whose countries are parallelograms is* 3*-colorable.*

Proof: The Erdős–de Bruijn theorem [**1**] states that an infinite map is 3-colorable whenever every finite submap is. Given a tidy finite map it suffices to show that it has a country having at most two neighbors. Recalling the Meisters two-ear theorem for polygons [**6**, p. 16], let us use the term *elbow* for such a country. Given an elbow, remove it, color the remainder by induction, and then replace the elbow and color it with the free color, which must exist since it has at most two neighbors. Thus we need to show only that every finite and tidy collection of parallelograms has an elbow. We do that in a separate lemma.

The Three-Elbow Lemma. *Any tidy, finite collection of parallelograms in the plane contains an elbow; if the collection has at least three elements then it must contain at least three elbows.*

Proof: The basic idea is that if there were no elbows then the boundary could never turn adequately in the right direction so as to close up. A checkerboard is a good example: the squares at the corners—where the boundary turns—are elbows. We now make this precise. Suppose, to get a contradiction, that we have a map as hypothesized with no elbows. We may work with a submap that is connected in the sense that there is a path from any country to any other that does not pass through any vertices of the map; such a submap cannot be disconnected by the removal of a single point. Let P be the simple, closed polygon that forms the exterior of the submap and let n be the number of vertices on P (Figure 2). The interior angles of P sum to $(n-2)\pi$. But at each vertex there is an edge coming in and an edge going out. The lack of elbows means that these edges belong to different countries. The parallelogram belonging to the edge coming in contributes two angles to the polygon's interior angles, and they sum to π. It follows that the sum of all the interior angles of P is at least $n\pi$, a contradiction. Since an elbow can subtract strictly less than π from this interior angle sum, there are at least three elbows. This succinct proof—a simplification of our original proof—is due to Michael Schweitzer (Berlin).

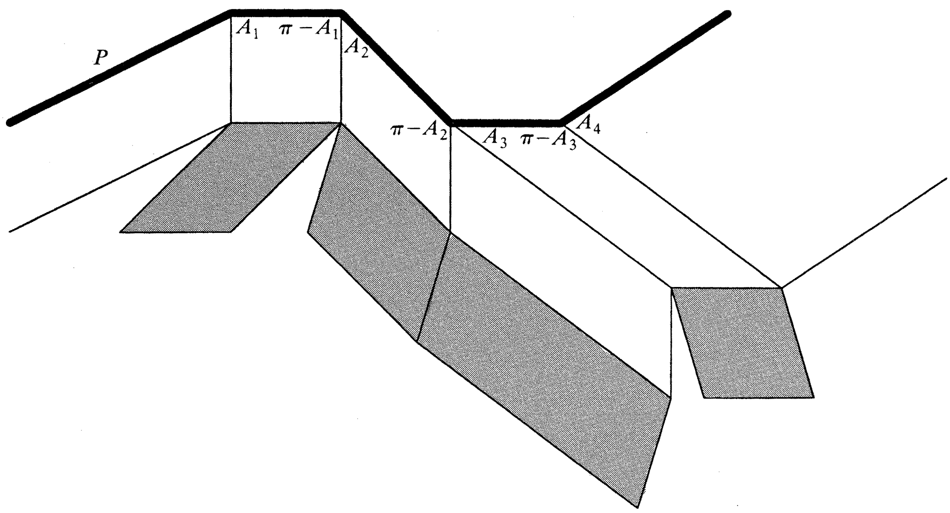

Figure 2. The thick line is part of the boundary of a tidy tiling by parallelograms: if there is no elbow, then each edge contributes π to the interior-angle sum.

The maps shown in Figure 3 ((a) is due to Rick Mabry) show that our results do not hold for convex quadrilaterals in general, or for untidy maps with parallelograms. Both maps require four colors. And these maps are minimal with respect to the number of countries used (thanks to David Castro for a proof of this for case (a)).

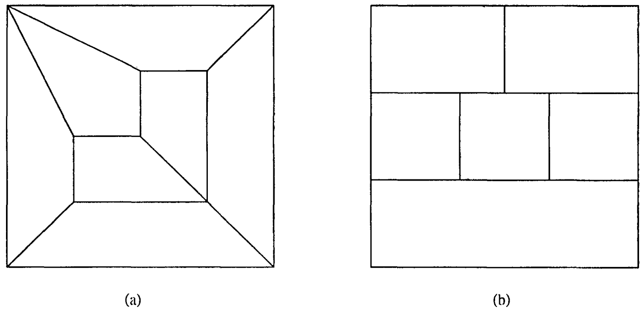

Figure 3. (a) A collection of 7 convex quadrilaterals that requires four colors. (b) An untidy rectangular tiling that needs four colors.

It is not at all clear whether these ideas yield anything for the Penrose kites and darts. Figure 4 shows a kite-and-dart configuration that is elbowless (there are many such); but it is easily 3-colored, and it seems as if larger kite-and-dart tilings can be 3-colored.

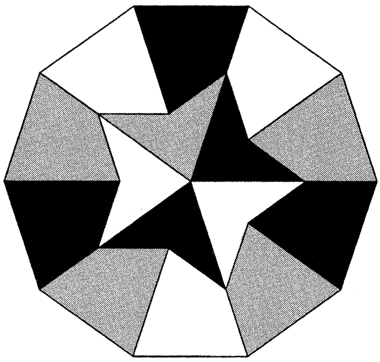

Figure 4. A collection of 15 Penrose kites and darts, each of which has degree 3 or 4.

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